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Kochen–Specker Sets and Generalized Orthoarguesian Equations

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Abstract. We prove that the 7oa class (equational variety) of generalized orthoarguesian lattices is properly included in all $n$oa classes for $n < 7$. This result strengthens the conjecture that any generalized orthoarguesian equation is strictly stronger than those of lower orders. The result emerged from our recent analysis of whether three-dimensional Kochen–Specker sets can be represented by Greechie lattices, which are a kind of orthomodular lattice.

1. Introduction

For a lattice to correctly represent a given formal description of a quantum system, it must at least satisfy all the equations satisfied by the lattice of all closed subspaces of a Hilbert space. In 1937, Husimi discovered that, for infinite-dimensional Hilbert spaces, this lattice satisfies the orthomodular law [1]. Since 1975, additional equations that it satisfies have been discovered. Among these, the only ones known that are directly related to the vector space of the underlying Hilbert space (i.e., excluding those that are related to states introduced on the lattice) are the generalized orthoarguesian equations (nOA, $n \geq 3$) [2]. Thus, these equations are an essential tool for analyzing lattices conjectured to represent particular experimental setups. If a lattice does not pass nOA for all n, then it is not a correct lattice [3].

In this paper, we show that Peres' Kochen–Specker (KS) set [4] can generate a set of lattices that violate the generalized orthoarguesian equation of order 7 but that satisfy the equations of orders 6 and less. This is achieved by considering only the orthogonality relations between Hilbert space vectors and ignoring any other relations between them. We describe a set of such orthogonality relations within Peres' setup by means of a so-called *Greechie lattice* (represented by a *Greechie diagram*). It turns out that "Peres' Greechie lattice" and numerous smaller Greechie lattices we can derive from it are counterexamples that prove the long-sought result that the 7OA equations is strictly stronger than the 6OA equation.

The goal of this paper is to show that Peres' Greechie lattice satisfies 3OA through 6OA but violates 7OA [in Sect. 3]. If we call the corresponding equational varieties (i.e., the classes of orthomodular lattices satisfying the equations) by 3oa through 7oa, this proves the series of proper inclusions 3oa $\supset$ 4oa $\supset$ 5oa $\supset$ 6oa $\supset$ 7oa. This follows from the result in this paper combined with our previous result, 3oa $\supset$ 4oa $\supset$ 5oa $\supset$ 6oa [5,6]. Finally, we show how we can modify Peres' Greechie lattice to generate simpler lattices with the same property, giving us counterexamples that are more practical to work with.

2. Lattice Definitions and Theorems

The closed subspaces of a Hilbert space, partially ordered by inclusion, form an algebra called a lattice [3]. Given a Hilbert space $\mathcal{H}$, we denote this algebra by $\mathcal{C}(\mathcal{H})$. In $\mathcal{C}(\mathcal{H})$, the operation *meet*, $a \wedge b$, corresponds to set intersection, $\mathcal{H}_a \bigcap \mathcal{H}_b$, of closed subspaces $\mathcal{H}_a, \mathcal{H}_b$ of Hilbert space $\mathcal{H}$, the ordering relation $a \le b$ corresponds to $\mathcal{H}_a \subseteq \mathcal{H}_b$, the operation *join*, $a \vee b$, corresponds to the smallest closed subspace of $\mathcal{H}$ containing $\mathcal{H}_a \bigcup \mathcal{H}_b$, and the *orthocomplement* $a^\perp$ corresponds to $\mathcal{H}_a^\perp$, the set of vectors orthogonal to all vectors in $\mathcal{H}_a$. Within Hilbert space, there is also an operation which has no parallel in $\mathcal{C}(\mathcal{H})$ (meaning that it cannot be defined in terms of lattice operations): the sum of two (not necessarily closed) subspaces $\mathcal{H}_a + \mathcal{H}_b$, which is defined as the set of sums of vectors from $\mathcal{H}_a$ and $\mathcal{H}_b$. We also have $\mathcal{H}_a + \mathcal{H}_a^\perp = \mathcal{H}$. One can define all the lattice operations on a Hilbert space itself following the above definitions ($\mathcal{H}_a \wedge \mathcal{H}_b = \mathcal{H}_a \bigcap \mathcal{H}_b$, etc.). Thus, we have $\mathcal{H}_a \vee \mathcal{H}_b = \overline{\mathcal{H}_a + \mathcal{H}_b} = (\mathcal{H}_a + \mathcal{H}_b)^{\perp\perp} = (\mathcal{H}_a^\perp \bigcap \mathcal{H}_b^\perp)^\perp$, [7, p. 175] where $\overline{\mathcal{H}_c}$ is the closure of $\mathcal{H}_c$ and, therefore, $\mathcal{H}_a + \mathcal{H}_b \subseteq \mathcal{H}_a \vee \mathcal{H}_b$. If $\mathcal{H}$ is finite-dimensional or if the closed subspaces $\mathcal{H}_a$ and $\mathcal{H}_b$ are orthogonal to each other, then $\mathcal{H}_a + \mathcal{H}_b = \mathcal{H}_a \vee \mathcal{H}_b$ [8, pp. 21–29], [1, pp. 66, 67], [9, pp. 8–16].

2.1. Lattice Definitions

We briefly recall the definitions we will need. For further information, see Refs. [5,6,10,11].

Definition 2.1. A *partial order* is a binary relation "$\le$" over a set P which is reflexive, antisymmetric, and transitive, i.e., for all a, b, and c in P, we have:

$$a \le a \qquad \text{(reflexivity);}$$
$$a \le b \quad \& \quad b \le a \quad \Rightarrow \quad a = b \qquad \text{(antisymmetry);}$$
$$a \le b \quad \& \quad b \le c \quad \Rightarrow \quad a \le c \qquad \text{(transitivity).}$$

A set with a partial order is called a *partially ordered set* (*poset*).

Definition 2.2. [12] A *lattice* (L) is an algebra $\langle \mathcal{L}_{\mathrm{O}}, \wedge, \vee \rangle$ such that the following conditions are satisfied for any $a, b, c \in \mathcal{L}_{\mathrm{O}}$: $a \vee b = b \vee a$, $a \wedge b = b \wedge a$, $(a \vee b) \vee c = a \vee (b \vee c)$, $(a \wedge b) \wedge c = a \wedge (b \wedge c)$, $a \wedge (a \vee b) = a$, $a \vee (a \wedge b) = a$.

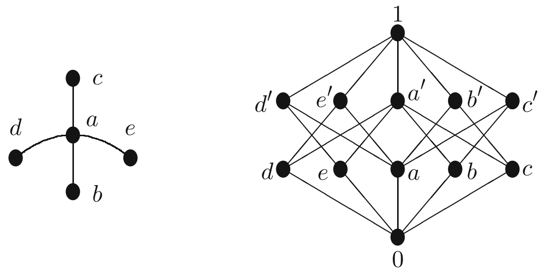

FIGURE 1. A Greechie diagram and its corresponding Hasse diagram

Theorem 2.3. [12] *The binary relation* $\leq$, *defined on an* L *as* $a \leq b \overset{\text{def}}{\Longleftrightarrow} a = a \wedge b$, *is a partial order. Thus every lattice is a poset.*

Definition 2.4. [13] An *ortholattice* (OL) is an algebra $\langle \mathcal{L}_{\mathrm{O}}, ^{\perp}, \wedge, \vee, 0, 1\rangle$ such that $\langle \mathcal{L}_{\mathrm{O}}, \wedge, \vee\rangle$ is a lattice with unary operation $^{\perp}$ called *orthocomplementation* which satisfies the following conditions for $a, b \in \mathcal{L}_{\mathrm{O}}$ ($a^{\perp}$ is called the *orthocomplement* of a): $a \vee a^{\perp} = 1$, $a \wedge a^{\perp} = 0$, $a \leq b \Rightarrow b^{\perp} \leq a^{\perp}$, $a^{\perp\perp} = a$.

Definition 2.5. [1] An *orthomodular lattice* (OML) is an OL in which the following condition (the *orthomodular law*) holds: $a \vee (a^{\perp} \wedge (a \vee b)) = a \vee b$.

As we shall see later (Theorem 2.16), in any OL the nOA law implies the orthomodular law.

Definition 2.6. A *Boolean algebra* (BA) is an OL in which the following condition (the *distributive law*) holds: $a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$.

2.2. Orthogonalities, Greechie Diagrams, and Greechie Lattices

Orthogonal vectors determine directions in which we can orient our detection devices and, therefore, also directions of observable projections. Subspaces (not necessarily closed) have a corresponding orthogonality relation denoted $\mathcal{H}_x \perp \mathcal{H}_y$, which is defined as $\mathcal{H}_x \subseteq \mathcal{H}_y^{\perp}$. We can choose one-dimensional subspaces $\mathcal{H}_a, \ldots, \mathcal{H}_e$ as shown in Fig. 1, where we denote them as $a, \ldots, e$. (Finite-dimensional subspaces are always closed and thus lattice elements of $\mathcal{C}(\mathcal{H})$.)

Definition 2.7. A *Hasse diagram* is a graphical representation of a poset where an element y is drawn above and connected to an element x if and only if $y > x$ and y is the least such element (i.e., y *covers* x).

Definition 2.8. In a poset with a least element 0, and *atom* is an element a that covers 0, i.e., there is no element b such that $0 < b < a$.

In the lattice $\mathcal{C}(\mathcal{H})$, an atom corresponds to a one-dimensional subspace of Hilbert space.

The orthogonality between subspaces—in our case corresponding to each chosen vector and a plane determined by the other two—can be seen in the

Hasse diagram shown on the right in Fig. 1. In particular, the orthogonalities are $a \perp b, c, d, e$ since $a \le b^\perp, c^\perp, d^\perp, e^\perp$, $b \perp c$ since $a \le c^\perp$, and $d \perp e$ since $d \le e^\perp$. Also, e.g., $b^\perp$ is the orthocomplement of b, corresponding to a plane to which any vector in b (and thus b itself) is orthogonal: $b^\perp = a \vee c$. Eventually, $b \vee b^\perp = 1$ where, in the case of $\mathcal{C}(\mathcal{H}), 1$ stands for $\mathcal{H}$.

Definition 2.9 (*Greechie diagram* [14]). The Hasse diagram for an OML consists of connected Hasse diagrams representing its maximal Boolean subalgebras, called *blocks*, and has a shorthand notation called a *Greechie diagram*. The notation represents the atoms within each block as dots connected by a line or smooth curve. The following conditions must be satisfied.

1. All blocks share a common 0 and 1.
2. If an atom a belongs to an intersection of blocks and, therefore, to both of them, then the blocks also share $a^\perp$;
3. Blocks contain three or more atoms.
4. Two blocks may not share more than one atom.

This definition is equivalent to Greechie's original definition [14]. Recently, the term Greechie diagram has been used to denote other kinds of hypergraphs related to pastings [15–17], Kochen–Specker sets [18], test spaces [19], etc. For these hypergraphs, condition 4 above does not necessarily hold, but for our elaboration and the generation of our diagrams it is essential. Since this condition is also present in the original definition, we embraced it.

Definition 2.10. A *loop* of order $n > 2$ is a set of blocks $B_1, \dots, B_n$ such that B_i shares an atom with B_{i+1} for $i < n$ and B_1 shares an atom with B_n.

Lemma 2.11. [14] *A Greechie diagram represents an orthomodular lattice if and only if the order of every loop of its blocks is at least* 5.

This lemma is known as the *Loop Lemma* [1, p. 38].

Definition 2.12. The unique orthomodular lattice represented by a Greechie diagram satisfying the Loop Lemma is called a *Greechie lattice*.

We stress here that the Loop lemma does not hold for lattices represented by the aforementioned pasting hypergraphs but only for the original Greechie diagrams and lattices as defined by Definition 2.9.

To write down a Greechie diagram as a string of characters, we adopt the following conventions.

We encode the atoms of a Greechie diagram (e.g., a, b, c, d, e in Fig. 1) by means of alphanumeric and other printable ASCII characters. Each vertex (atom) is represented by one of the following characters: 1 2 3 4 5 6 7 8 9 A B C D E F G H I J K L M N O P Q R S T U V W X Y Z a b c d e f g h i j k l m n o p q r s t u v w x y z ! " # $ % & ' ( ) * - / : ; < = > ? @ [\ ] ^ _ ` { | } ~ , and then again all these characters prefixed by '+', then prefixed by '++', etc. There is no upper limit on the number of atoms that can be represented.

Each block (continuous line connecting dots in a Greechie diagram) is represented by a string of characters that represent atoms. Blocks are separated by commas. The order of the blocks is irrelevant, however, we shall often present them starting with blocks forming the biggest loop to facilitate their possible drawing. A string ends with a full stop (i.e., a period). Skipping of characters is allowed.

2.3. Generalized Orthoarguesian Equations

The *generalized orthoarguesian equations* nOA [5,6], which hold in the lattice $\mathcal{C}(\mathcal{H})$, follow from the following set of equations.

Theorem 2.13. *Let $\mathcal{M}_0,\dots,\mathcal{M}_n$ and $\mathcal{N}_0,\dots,\mathcal{N}_n, n\ge 1$, be any subspaces (not necessarily closed) of a Hilbert space, and let $\bigcap$ denote the set-theoretical intersection of two subspaces and $+$ their subspace sum. We define the subspace term $\mathcal{T}_n(i_0,\dots,i_n)$ recursively as follows, where $0\le i_0,\dots,i_n\le n$:*

$$\mathcal{T}_1(i_0,i_1)=(\mathcal{M}_{i_0}+\mathcal{M}_{i_1})\bigcap(\mathcal{N}_{i_0}+\mathcal{N}_{i_1}) \tag{2.1}$$

$$\begin{aligned}\mathcal{T}_m(i_0,\dots,i_m)&=\mathcal{T}_{m-1}(i_0,i_1,i_3,\dots,i_m)\\ &\bigcap(\mathcal{T}_{m-1}(i_0,i_2,i_3,\dots,i_m)+\mathcal{T}_{m-1}(i_1,i_2,i_3,\dots,i_m)),\quad 2\le m\le n\end{aligned} \tag{2.2}$$

For $m=2$, this means $\mathcal{T}_2(i_0,i_1,i_2)=\mathcal{T}_1(i_0,i_1)\bigcap(\mathcal{T}_1(i_0,i_2)+\mathcal{T}_1(i_1,i_2))$. Then, the following condition holds in any finite- or infinite-dimensional Hilbert space for $n\ge 1$:

$$(\mathcal{M}_0+\mathcal{N}_0)\bigcap\cdots\bigcap(\mathcal{M}_n+\mathcal{N}_n)\subseteq\mathcal{N}_0+(\mathcal{M}_0\bigcap(\mathcal{M}_1+\mathcal{T}_n(0,\dots,n))). \tag{2.3}$$

Proof. As given in [20,3] □

We will use the above theorem to derive a condition that holds in the lattice of closed subspaces of a Hilbert space. In doing so, we will make use of the definitions introduced above and the following well-known [8, p. 28] lemma.

Lemma 2.14. *Let $\mathcal{M}$ and $\mathcal{N}$ be two closed subspaces of a Hilbert space. Let $\vee$ denote the join of two subspaces (as defined at the beginning of Sect. 2). Then*

$$\mathcal{M}+\mathcal{N}\subseteq\mathcal{M}\vee\mathcal{N} \tag{2.4}$$

$$\mathcal{M}\perp\mathcal{N}\Rightarrow\mathcal{M}+\mathcal{N}=\mathcal{M}\vee\mathcal{N} \tag{2.5}$$

Theorem 2.15 (Generalized Orthoarguesian Laws). *Let $\mathcal{M}_0,\dots,\mathcal{M}_n$ and $\mathcal{N}_0,\dots,\mathcal{N}_n, n\ge 1$, be closed subspaces of a Hilbert space. We define the term $\mathcal{T}_n^\vee(i_0,\dots,i_n)$ by substituting $\vee$ for $+$ in the term $\mathcal{T}_n(i_0,\dots,i_n)$ from Theorem 2.13. We also substitute $\wedge$ for $\bigcup$ and $\le$ for $\subseteq$, since these are equivalent as described at the beginning of Sect. 2. Then, the following condition holds in any finite- or infinite-dimensional Hilbert space for $n\ge 1$:*

$$\begin{aligned}&\mathcal{M}_0\perp\mathcal{N}_0\ \&\ \cdots\ \&\ \mathcal{M}_n\perp\mathcal{N}_n\\ &\quad\Rightarrow(\mathcal{M}_0\vee\mathcal{N}_0)\wedge\cdots\wedge(\mathcal{M}_n\vee\mathcal{N}_n)\le\mathcal{N}_0\vee(\mathcal{M}_0\wedge(\mathcal{M}_1\vee\mathcal{T}_n^\vee(0,\dots,n))).\end{aligned} \tag{2.6}$$

Proof. By the orthogonality hypotheses and Eq. (2.5), the left-hand side of Eq. (2.6) equals the left-hand side of Eq. (2.3). By Eq. (2.4), the right-hand side of Eq. (2.3) is a subset of the right-hand side of Eq. (2.6). Equation (2.6) follows by Theorem 2.13 and the transitivity of the subset relation. □

Theorem 2.16. *An* OL *in which Eq.* (2.6) *holds is an* OML.

Proof. It suffices to show this for the lowest-order equation, which follows from the higher order ones. For $n = 1$, we can express Eq. (2.6) as

$$x \perp y \ \&\ z \perp w \Rightarrow (x \vee y) \wedge (z \vee w) \leq y \vee (x \wedge (z \vee ((x \vee z) \wedge (y \vee w)))). \quad (2.7)$$

Putting $b, 0, a, a^\perp$ for x, y, z, w respectively, the hypotheses are satisfied and the conclusion becomes $(b \vee 0) \wedge (a \vee a^\perp) \leq 0 \vee (b \wedge (a \vee ((b \vee a) \wedge (0 \vee a^\perp)))))$. Simplifying, we get $b \leq b \wedge (a \vee (a^\perp \wedge (a \vee b)$. Dropping the conjunct b from the right-hand side, adding the disjunct a to the left-hand side, and noticing that the other direction of the resulting inequality holds in any OL, we arrive at $a \vee b = a \vee (a^\perp \wedge (a \vee b))$, which is the orthomodular law (Definition 2.5). □

We mention that the orthomodular law also follows (in any OL) from the nOA laws in the form of Eq. (2.9) below. However, those equations make use of the orthomodular law for their derivation from Eq. (2.6). The above theorem gives us an alternate way to derive the orthomodular law directly from Hilbert space that is, in some ways, more elementary than the traditional proof by contradiction (e.g., Ref. [1, p. 65]).

Reference [5] shows that in any OML, Eq. (2.6) is equivalent to the $(n + 2)$OA law Eq. (2.9), thus establishing the proof of Theorem 2.18.

Definition 2.17. We define an operation $\overset{(n)}{\equiv}$ on n variables $a_1, \ldots, a_n$ $(n \geq 3)$ as follows:

$$a_1 \overset{(3)}{\equiv} a_2 \overset{\text{def}}{=} ((a_1 \to a_3) \wedge (a_2 \to a_3)) \vee ((a_1^\perp \to a_3) \wedge (a_2^\perp \to a_3))$$
$$a_1 \overset{(n)}{\equiv} a_2 \overset{\text{def}}{=} (a_1 \overset{(n-1)}{\equiv} a_2) \vee ((a_1 \overset{(n-1)}{\equiv} a_n) \wedge (a_2 \overset{(n-1)}{\equiv} a_n)), \qquad n \geq 4. \quad (2.8)$$

The operation $a \to b$ is defined as $a^\perp \vee (a \wedge b)$. In the transition from $n - 1$ to n, the hidden implicit variables in the notation are *not* renamed. For a worked-out example of this notation, the reader can consult the footnote to Def. 5.1 in Ref. [20].

Theorem 2.18. *The* nOA *laws*

$$(a_1 \to a_3) \wedge (a_1 \overset{(n)}{\equiv} a_2) \leq a_2 \to a_3. \quad (2.9)$$

hold in the lattice $\mathcal{C}(\mathcal{H})$.

The class of equations (2.9) is the *generalized orthoarguesian equations* nOA. [5,6]

3. Main Result: Lattices That Satisfy 6OA and Violate 7OA

The KS theorem claims that experimental recordings cannot be predetermined, i.e., fixed in advance. Its best known proof is based on sets (KS sets) to which

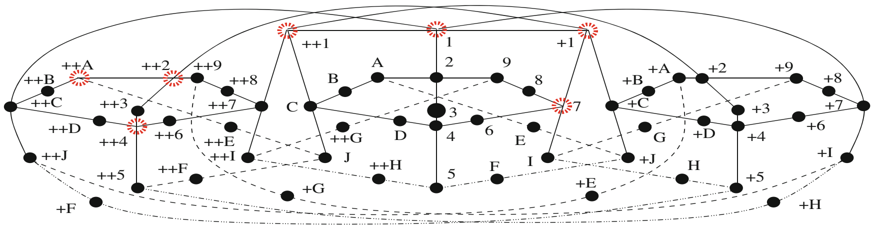

FIGURE 2. Peres' Greechie lattice. *Red* (*online*) *rings* denote atoms at which Peres' lattice violates 7OA, i.e., the failing assignment of atoms and co-atoms to the variables of 7OA in the form of Eq. (2.9)

it is impossible to ascribe classical 0-1 values. One of them is Peres' set (Fig. 2). It has 57 vectors and 40 triads of mutually orthogonal vectors.

As explained in Sect. 2.2, three atoms in a block of a three-dimensional Greechie diagram are orthogonal to each other. Also, any three vectors in a triad in Peres' set are orthogonal to each other. Since every loop made of Peres' triads is of order at least 5, we build a Greechie diagram with the idea of obtaining a Greechie lattice. We label a block in the diagram according to labels in Peres' set and we label each atom of the diagram according to labels of Peres' vectors. This Peres' Greechie diagram can be written as: `123, 345, 467, 789, 92A, ABC, CD4, AE+J, 5F+J, IG+9, IH+5, I7+1, JC++1, ++1+2+3, +3+4+5, +4+6+7, +7+8+9, +9+2+A, +A+B+C, +C+D+4, +A+E++J, +5+F++J, +I+G++9, +I+71, +I+H++5, +J+C+1, +1++2++3, ++3++4++5, ++4++6++7, ++7++8++9, ++9++2++A, ++A++B++C, ++C++D++4, ++A++EJ, ++5++FJ, ++I++G9, ++I++7++1, ++I++H5, ++J++C1, 1+1++1.`

Since now every loop made of this Peres' Greechie diagram is also of order at least 5, according to Lemma 2.11, it represents a Greechie lattice. We call it *Peres' Greechie lattice*. We stress that the Peres' Greechie lattice is *not* a lattice that corresponds to the $\mathcal{C}(\mathcal{H})$ lattice of a full Hilbert space description of Peres' set. The only thing these two lattices have in common are the atoms of their respective Hasse diagrams. Peres' Greechie lattice is not even a subalgebra of the latter lattice [3].

When we check—by our program `latticeg` described in Sect. 4—whether Peres' Greechie lattice satisfies the nOA equations, we find out that it satisfies 3OA through 6OA but violates 7OA at `++1, ++4, 1, 7, +1, ++A, ++23`, which we indicated with the help of rings in Fig. 2. Now, we show how to arrive at much smaller lattices that also satisfy 6OA and violate 7OA. The procedure makes use of the program `latticeg` to eliminate atoms and blocks that did not take part in the violations of 7OA we originally found.

When we apply `latticeg` to the equation 7OA and it arrives at atoms (or more precisely, lattice nodes) at which 7OA fails, the program gives the nodes we listed above, and it also gives us the following additional information about the failure:

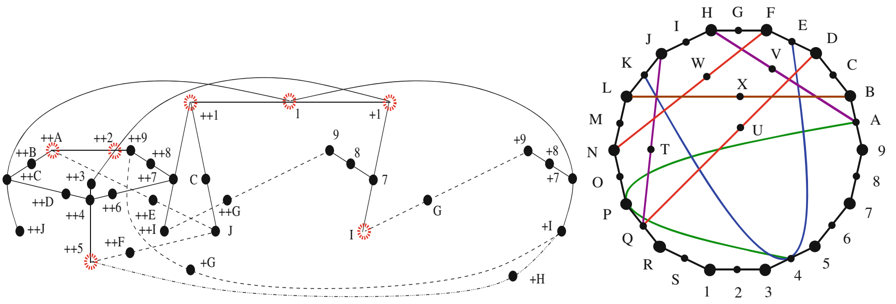

Figure 3. A lattice with 33 atoms and 21 blocks that satisfies 6OA and violates 7OA. *Red* (*online*) *rings* show atoms that take part in a violation of 7OA. The *left* and *right* diagrams are isomorphic to each other (i.e., are two ways of drawing the same lattice)

```
Greechie atoms not visited: 2 3 4 ...
Greechie blocks that do not affect the failure: 345 ABC CD4 ...
```

If, during the evaluation of the failing assignment, the meets and joins contained in a block are never used, then that block is unrelated to the failure. The program accumulates such blocks and puts them into a list called "don't affect the failure" as illustrated by the sample printout above. After removing these from the Peres' Greechie lattice of Fig. 2 and renaming the atoms, we end up with the smaller Greechie lattice 123, 345, 567, 789, 9AB, BCD, DEF, FGH, HIJ, JKL, LMN, NOP, PQR, RS1, 4EK, 4AP, AVH, BXL, DUQ, FWN, JTQ which is shown in Fig. 3. The left figure shows the blocks we dropped from Fig. 2, and the right one is given in the representation we previously used to show violations of 3OA through 6OA at lattices presented in [2,6,20] with the maximal loop (tetrakaidecagon, 14-gon) it contains.

A set of lattices between Peres' 57-40 and the 33-21 shown here can be obtained by adding to the 33-21 lattice any of the blocks removed from the 57-40, giving us $2^{40-21} = 2^{19}$ lattices altogether. All of these violate 7OA, because the removed blocks do not participate in the 7OA failure we observed. It is expected that most or all of these lattices will satisfy 6OA, which would provide many additional counterexamples, if they are desired, that will distinguish the two equations. (Our observation has been that in most cases, if a lattice satisfies an equation, it will continue to satisfy it when a block is removed. But since removing a block does not necessarily create a sublattice, there are rare exceptions [3].)

4. Algorithms and Programs

The main program that we used for this work was `latticeg`, which is a general-purpose utility for testing equations against orthocomplemented lattices expressed in the form of Greechie diagrams. Its algorithm is described in Ref. [21].

The nOA law in the form derived directly from Hilbert space, Eq. (2.6), has $2n-2$ variables, whereas in the equivalent form of Eq. (2.9) it has n variables. Since testing an equation with m variables against a lattice with k nodes requires that up to k^m combinations be checked, it is more efficient to use the form of Eq. (2.9).

Equation (2.9) has $8\cdot 3^{n-3}+4$ occurrences of its n variables. For faster computation, we found an equivalent with $6\cdot 3^{n-3}+3$ variable occurrences (which equals 166 for 6OA and 489 for 7OA). The following theorem shows this equivalent form for $n=3$. The proof is similar for larger n. The general form for larger n can be inferred by looking at the proof, although we have not defined a "compact" notation for it as we have for Eq. (2.9).

Theorem 4.1. *An* OML *in which the equation*

$$a\wedge((a\wedge b)\vee((a\to c)\wedge(b\to c)))\le b^\perp\to c \tag{4.1}$$

holds is a 3OA *and vice-versa.*

Proof. **For Eq. (4.1):** To obtain the 3OA law, Eq. (2.9), from Eq. (4.1), we substitute $a\to c$ for a and $b\to c$ for b, then we use the OML identities $(a\to c)\to c=a^\perp\to c$, $(b\to c)\to c=b^\perp\to c$, and $(b^\perp\to c)\to c=b\to c$.

For the converse, since $x\le x^\perp\to y$,

$$\begin{aligned}
&a\wedge((a\wedge b)\vee((a\to c)\wedge(b\to c)))\\
&\quad\le(a^\perp\to c)\wedge(((a^\perp\to c)\wedge(b^\perp\to c))\vee((a\to c)\wedge(b\to c)))\\
&\quad=(a^\perp\to c)\wedge(a^\perp\overset{(3)}{\equiv}b^\perp)\\
&\quad\le b^\perp\to c,
\end{aligned}$$

where for the last step we used an instance of Eq. (2.9) for $n=3$. □

Because of the large size of the nOA equations for larger n, in order to ensure that our input to `latticeg` was free from typos we used an auxiliary utility program, `oagen`, to generate nOA equations in the form of either Eq. (2.9) or Eq. (4.1).

The evaluation of the 7OA equation on the Peres Greechie diagram involves 7 nested loops, each with 116 iterations (since its Hasse diagram has 116 nodes). For the shorter equation of the form of Eq. (4.1), each evaluation at the innermost loop involves an assignment to 489 variable occurrences and 487 join, meet, and $\to$ operations (the last having a precomputed table in memory from its join, meet, and orthocomplementation expansion). Thus, $116^7\cdot 489=138,202,145,015,414,784$ (138 quadrillion) operation evaluations ($489=487+1+1$ includes the final $\le$ comparison and a single orthocomplementation) are required for a full scan.

Such a direct, full evaluation is a challenge on today's hardware, even with a cluster of processors, unless one is very lucky to encounter a failure early on in the scan (and we were). In addition, we made several enhancements to `latticeg` to help make this project more feasible:

- The main algorithm was improved. The original algorithm assigned each possible combination of lattice nodes to the equation variables, then evaluated the resulting equation according to the structure of the lattice (i.e., the suprema, infima, and orthocomplements in the Hasse diagram derived from the input Greechie diagram). The main scan consists of nested loops that processes all nodal assignments to the first variable in the outermost loop, then all assignments to the second variable in the next inner loop, and so on. Since it has seven variables, testing the 7OA equation involves 7 nested loops.

  The new algorithm takes into account, at each loop level, the variables in outer loops (which have known assignments) and evaluates as much of the equation as it can with those known variables. The equation is then shrunk with these partial evaluations, for further processing at that and deeper loop levels. Eventually, the equation is shrunk to a length of one, which means that it is completely evaluated. While a length of one will always be obtained at the innermost loop level, it may also occur at an outer level (such as when an expression containing not-yet-assigned variables is conjoined with a partial evaluation that resulted in lattice 0). In such cases, processing of further inner loops becomes unnecessary. So, the new algorithm benefits from (1) shorter equations to evaluate at deeper loop levels and (2) possible skipping of the deepest loops. Overall, this results in an empirical speedup of about a factor of 10 for the 7OA equation evaluation.

  Because of the complexity of the new partial evaluation algorithm, it was put into a new version of `latticeg` called `lattice2g`. This allows us to check that the old and new algorithms produce the same result, helping to make sure there is not a program bug in the new algorithm. Having two programs also allow us to directly measure the speedup afforded by the new algorithm.
- For testing a huge lattice, a feature was added to break up the testing into several independent parts. In this way, the different parts can be run on different processors in our cluster. The test can be partitioned into any number of outermost and first inner loop iterations. For example, the Peres' Greechie diagram has a Hasse representation with 116 nodes. We can specify, e.g., that the cluster test the 98th iteration (out of 116) of the outmost loop and the 101st through 110th iteration (out of 116) of the next inner loop.
- A feature was added to analyze an equation failure to determine what nodes, atoms, and blocks were not involved in the failure. In particular, a block is said not to affect the failure whenever all operations that "visit" (non-0 and non-1) nodes in the block do not involve any other (non-0 and non-1) nodes in that block. This is described in more detail in Sect. 3, where we show how this feature was used to determine which blocks could be removed from Peres' Greechie lattice to obtain a smaller lattice that satisfies 6OA but violates 7OA

5. Conclusion

After 65 years of research carried out in the field of the algebraic structure underlying quantum Hilbert space—the Hilbert lattice—only one equation (beyond the orthomodular lattice laws) that holds in it was found: the orthoarguesian equation. Some equivalent forms and consequences of the orthoar guesian equation, which collectively we will call OAs, were found in the 1980s and 1990s. All other equations known to hold in $\mathcal{C}(\mathcal{H})$ require a state introduced onto the lattice elements.

Then in 2000, we found [5] a class ($n$oa) of lattices determined by *generalized orthoarguesian equations* (nOA) and proved that the following inclusion holds: $n$oa $\supseteq$ $(n{+}1)$oa. We also proved that all previously found OAs are equivalent to either 3OA or 4OA, we proved that 4OA is strictly stronger than 3OA, and we found lattices in which 4OA passed but 5OA failed and (after much computational work) lattices in which 5OA passed and 6OA failed. [6]

In this paper, we found a set of lattices—shown in Figs. 2 and 3 and obtained as explained in Sect. 3—in which 6OA passes and 7OA fails.

Because we do not have a proof for the conjecture that the inclusion $n$oa $\supset$ $(n+1)$oa is strict for all n, each new counterexample, especially for small n, provides important additional evidence.

The new counterexample is also important because it provides an additional lattice in the sequence of counterexamples. Finding a pattern in this sequence—which is an ongoing project, as we investigate features such as common isomorphic subgraphs and the details of failures—may provide an important clue for arriving at a general proof, such as one by induction. We point out that the numbers of elements (atoms and blocks) of the smallest known lattices that satisfy nOA but violate $(n+1)$OA do not appear to grow exponentially. For $3 \le n \le 7$ we have 13, 17, 22, 28, 33 and 7, 10, 13, 18, 21 atoms and blocks, respectively [6].

We obtained the lattices in this paper by analyzing three-dimensional Peres' Kochen–Specker set. In three-dimensional Hilbert space, a correspondence between Kochen–Specker sets and Greechie diagrams can be established. We scanned over 10,000 KS Greechie diagrams, and they all violated 3OA except Peres'. When we reached the result that it satisfied 6OA we wanted to see whether it would violate 7OA. The verification turned out to be extremely demanding because of the number of terms 7OA consists of, and we had to design a number of algorithms and programs for the task. The algorithms and programs are described in Sect. 4. The task ran over a month on a 500 CPU cluster.

And indeed we found that the Peres' Greechie lattice that corresponds to Peres' Kochen–Specker set can serve as a counterexample for the above proof (see Fig. 2). It also served as a generator for smaller counterexamples we described in Sect. 3, the smallest of which is shown in Fig. 3.

An open question is what additional conditions must be added to the nOA equations to specify $\mathcal{C}(\mathcal{H})$, for both the finite and the infinite dimensional cases? Are there other classes of lattice equations that hold in $\mathcal{C}(\mathcal{H})$

when we do not introduce states on it? (The other known equations such as Godowski's and Mayet's [6] assume states.) How far can we define $\mathcal{C}(\mathcal{H})$ only by means of sets of equations added to an OL?

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