Foundations of Physics, Vol. 32, No. 3, March 2002 (© 2002)

Deduction, Ordering, and Operations in Quantum Logic

Normal D. Megill[1] **and Mladen Pavičić**[2]

Received August 24, 2001; revised October 15, 2001

We show that in quantum logic of closed subspaces of Hilbert space one cannot substitute quantum operations for classical (standard Hilbert space) ones and treat them as primitive operations. We consider two possible ways of such a substitution and arrive at operation algebras that are not lattices what proves the claim. We devise algorithms and programs which write down any two-variable expression in an orthomodular lattice by means of classical and quantum operations in an identical form. Our results show that lattice structure and classical operations uniquely determine quantum logic underlying Hilbert space. As a consequence of our result, recent proposals for a deduction theorem with quantum operations in an orthomodular lattice as well as a, substitution of quantum operations for the usual standard Hilbert space ones in quantum logic prove to be misleading. Quantum computer quantum logic is also discussed.

KEY WORDS: orthomodular lattice; quantum logic; quantum conjunction; quantum disjunction; deduction theorem; ordering relation; operation algebras; quantum computer.

1. INTRODUCTION

Quantum computer theory recently introduced quantum logic as an algebra of quantum bits (qubits) handled by quantum logic gates[(1)] and as a quantum counterpart of the classical Boolean algebra which is an algebra

[1] Boston Information Group, 30 Church St., Belmont, Massachusetts 02478; e-mail: nm@alum.mit.edu; web page: http://www.metamath.org

[2] University of Zagreb, AGG Fakultet, Physics Division, Kačićeva 26, HR-10000 Zagreb, Croatia; e-mail: mpavicic@faust.irb.hr; web page: http://m3k.grad.hr/pavicic

0015-9018/02/0300-0357/0

of classical bits handled by classical logic gates in today's computers. This quantum algebra is still not well defined but it may eventually emerge as a variety of equations defining quantum logic of Hilbert space—so-called Hilbert lattice (algebra of subspaces or, equivalently, operators of Hilbert space)—introduced half a century ago.[2, 3] A project of reducing the two logics to each other should first of all address those features of the Hilbert space quantum logic whose implementation into the computer quantum logic are hardly possible or problematic. Such features are:

- The Hilbert space quantum logic (Hilbert lattice) is (ortho)isomorphic to the lattice of the closed subspaces of any Hilbert space. It has recently been shown that *infinite orthogonality* and *unitarity* added to the lattice make sure that the field over which any infinite dimensional Hilbert space is defined must be either complex, or real or quaternionic. This does not hold for a finite dimensional Hilbert space which allows non-standard (e.g., Keller's[4]) fields that can make the space nonarchimedean. The latter possibility should be eliminated in favor of a complex field, but this requires further research since the theory of finite Hilbert lattices and of finite complex Hilbert space is poorly developed. Quantum computer quantum logic obviously must be finite dimensional and therefore we need a finite Hilbert lattice theory with conditions that eliminate exotic properties and have a plausible physical interpretation. Besides, a finite dimensional quantum theory might offer us a model for discrete space which goes beyond any numerical grid approximation of quantum state equations (as given, e.g., by Ref. 5 for quantum computer).

- Hilbert lattice theory contains conditions of the second order which involve universal and existential quantifiers. Whether they can be translated into algebraic conditions and equational series, and as such possibly handled and approximated by quantum computer, is an open question. An algebraic approach to the Hilbert lattices, whose goal is to substitute the aforementioned quantifier conditions with algebraic equations has been started only recently.[6–9]

- There was a long standing question on whether a *proper* logic—not its above considered algebraic model which is misleadingly called quantum *logic*—underlying Hilbert space can play the same role classical logic, underlying the Boolean algebra, played—it was believed—in any classical physical and computer theory.[10–17] However, two years ago it was discovered[18] that there are *proper* logics for neither quantum nor classical theories. It turned out that both quantum and classical *proper* logics have at least two ortholattice models each and that their syntax correspond to models that are neither Hilbert lattice nor Boolean algebra, respectively.

Hence, *proper* logics can characterize neither quantum nor classical computers. More generally, they can characterize neither quantum nor classical theories— physical as well as mathematical. For, obviously, one cannot claim that a statement holds in *proper* quantum logic if and only if it is true in orthomodular lattices and at the same time that a statement holds in *proper* quantum logic if and only if it is true in non-orthomodular lattices. The same holds for *proper* classical logic. Consequently, another long standing *proper* quantum logic issue— whether or not there is a *deduction theorem* for the logic—is put aside by the above result. However, the deduction theorem was recently reformulated in a lattice theoretic framework and we will consider the latter reformulation in the present paper. This will take us to a problem of whether there is *quantum* lattice ordering in Hilbert lattices.

- Orthomodular lattices—which contain Hilbert lattices—use operations taken over from Hilbert space. However, there are also five the so-called quantum versions of them.[19] The latter reduce to the former for compatible variables. For almost half a century various authors have tried to find *proper* quantum operations. This will also take us to a problem of possible *quantum* ordering in Hilbert lattices. It will turn out that in a standard formulation of Hilbert lattices with standard operations there are *quantum* orderings. But if we wanted to use only *quantum* operations and only *quantum* orderings, we would arrive at algebras that are not lattices and which therefore cannot be made (by means of additional conditions) orthoisomorphic to the lattice of closed subspaces of Hilbert space. Hence, the standard lattice operations inherited from Hilbert space are the only "proper" operations. The result greatly simplifies approaches to the first two points above since it shows that any use of "quantum operations" in finding new conditions and equations in Hilbert lattices can only be a casual matter of convenience. E.g., in formulations of orthoarguesian laws some of quantum operations appear in characterizations of equations[9] and now we know that they only make equations shorter to write.

Taken together, setting down the problems with quantum operations, ordering, and deduction will narrow the gap between the two logics and this is what we strive at in this paper. However, before dwelling to the task, the following historical comments might be helpful.

Thirty years ago Finch[20] noticed that, in an orthomodular lattice, the operation $a \cap (b \cup a')$, which we denote as $a \cap_1 b$ and call *quantum conjunction*, "could be interpreted as an operation of logical conjunction" and that it satisfies the following condition: $b \cap_1 a \leqslant c \Leftrightarrow a \leqslant b \rightarrow_1 c$ where $a \rightarrow_1 b$ denotes $(a \cap_1 b')'$ and is called *quantum implication*. Román and Rumbos[21] tried to give a meaning to quantum operations and Román and

Zuazua claim that Finch's condition "is simply *some* kind of *deduction theorem*.[22]" The condition really shows a striking similarity with what one could call a *deduction theorem in a Boolean algebra*: $b \cap a \leqslant c \Leftrightarrow a \leqslant b \to c$ [where $a \to b$, called *standard implication*, denotes $(a \cap b')'$] and at the same time is in an apparent striking contradiction with a previous result by Malinowski according to which "no consequence operation determined by any class of orthomodular lattices admits the deduction theorem.[23]" On the other hand, there are authors who propose that we use *quantum operations* as "especially interesting from a physical point of view,[24]" Many others considered particular quantum operations according to various properties they judged such an operation should have.[25–27, 14, 15] Because the whole issue is of a general interest for characterizations of algebras underlying a Hilbert space formulation of quantum mechanical systems, in this paper we investigate whether it is possible to formulate orthomodular lattice by means of non-standard "quantum" operations.

As we show in the next sections, it turns out that all previous authors make use of the ordering relation $\leqslant$ from the standardly defined ortholattice. This ordering relation is defined by means of the standard conjunction operation: $a \leqslant b \overset{\text{def}}{\Leftrightarrow} a \cap b = a$ and not by means of quantum operations. However, if we wanted to switch from the standard operations to quantum ones, we should redefine $\leqslant$ as well.[19] In Sec. 2 we do so and show that *quantum operations* alone cannot serve us for defining quantum logic. In Sec. 3 we construct an algebra based on the aforementioned quantum conjunction $\cap_1$ so as to contain both a quantum ordering and a deduction theorem, but which is not a lattice—although one can embed an orthomodular lattice in it. In Secs. 3 and 6 we generalize the result to algebras based on quantum disjunctions $\cup_i$, $i = 2,\ldots, 5$. In Secs. 3–5 we give an algebra which for both classical and quantum operations of quantum conjunction and disjunction have identical structural forms for all its equations. Again, the resulting algebra is not a lattice although one can embed an orthomodular lattice in it.

2. DEDUCTION AND ORDERING

Let us first define an ortholattice with the help of the lattice ordering relation as follows.[28]

Definition 2.1. An ortholattice, OL is an algebra $\langle \mathcal{OL}_0, ', \cup, \cap \rangle$ such that the following conditions are satisfied for any $a, b, c \in \mathcal{OL}_0$:

OL1	$a \leqslant a$
OL2	$a \leqslant b \quad \& \quad b \leqslant a \quad \Rightarrow \quad a = b$
OL3	$a \leqslant b \quad \& \quad b \leqslant c \quad \Rightarrow \quad a \leqslant c$
OL4	$a \leqslant a \cup b \quad \& \quad b \leqslant a \cup b$
OL5	$a \leqslant c \quad \& \quad b \leqslant c \quad \Rightarrow \quad a \cup b \leqslant c$
OL6	$a \leqslant b \cup b'$
OL7	$a = a''$
OL8	$a \leqslant b \quad \Rightarrow \quad b' \leqslant a'$
OL9	$a \cap b = (a' \cup b')'$

where

$$a \leqslant b \overset{\text{def}}{\Leftrightarrow} a \cup b = b \tag{2.1}$$

In addition, since $a \cup a' = b \cup b'$ for any $a, b \in \mathcal{O}\mathcal{L}_0$, we define:

$$1 \overset{\text{def}}{=} a \cup a' \tag{2.2}$$

$$0 \overset{\text{def}}{=} a \cap a' \tag{2.3}$$

This definition will prove itself convenient and economic, for yielding our results, but since it is not widely known we shall first show its equivalence to the following standard definition.[29]

Lemma 2.2. The above definition is equivalent to the following standard one.

A lattice, L is an algebra $\langle \mathcal{L}_0, \cup, \cap \rangle$ such that the following conditions are satisfied for any $a, b, c \in \mathcal{L}_0$:

L1a	$a \cup b = b \cup a$
L1b	$a \cap b = b \cap a$
L2a	$(a \cup b) \cup c = a \cup (b \cup c)$
L2b	$(a \cap b) \cap c = a \cap (b \cap c)$
L3a	$a \cup (a \cap b) = a$
L3b	$a \cap (a \cup b) = a$

where $a \leqslant b \overset{\text{def}}{\Leftrightarrow} a \cup b = b$.

An ortholattice, OL is an algebra $\langle \mathcal{OL}_0, \cup, \cap, ', 0, 1\rangle$ such that $\langle \mathcal{OL}_0, \cup, \cap\rangle$ is a lattice and in which the unary operation $'$ and the constants 0, 1 satisfy the following conditions for any $a, b \in \mathcal{OL}_0$:

OL1a′	$a \cup a' = 1$
OL1b′	$a \cap a' = 0$
OL2′	$a = a''$
OL3′	$a \leqslant b \Rightarrow b' \leqslant a'$

Proof. That Definition 2.1 follows from the standard definition is well-known. In particular, OL1 through OL5 follow from the conditions for L only.

The proof of the other direction is slightly less obvious, so, we show how to prove L1a, L2a, and L3a. L1b, L2b, and L3b are duals that follow using OL9 and OL7.

L1a follows from OL4, OL5, and OL2.

To prove L2a we first apply OL4 twice and from OL3 we get $a \leqslant (a \cup b) \cup c$, $b \leqslant (a \cup b) \cup c$, and $c \leqslant (a \cup b) \cup c$. Applying OL5 first to the last two equations and then reapplying OL5 yields $a \cup (b \cup c) \leqslant (a \cup b) \cup c$. Analogously, we get $(a \cup b) \cup c \leqslant a \cup (b \cup c)$. OL2 then yields L2a.

To prove L3a we first obtain $a \cap b \leqslant a$ from OL4, OL8, OL9, and OL7. From this, OL1 and OL5 yield $a \cup (a \cap b) \leqslant a$. The other direction follows from OL4, and OL2 finally yields L3a. □

Next, we introduce the following additional operations:

Definition 2.3. We define quantum implications as

$$a \to_1 b \overset{\text{def}}{=} a' \cup (a \cap b) \tag{2.4}$$

$$a \to_2 b \overset{\text{def}}{=} b' \to_1 a' \tag{2.5}$$

$$a \to_3 b \overset{\text{def}}{=} ((a' \cap b) \cup (a' \cap b')) \cup (a \cap (a' \cup b)) \tag{2.6}$$

$$a \to_4 b \overset{\text{def}}{=} b' \to_3 a' \tag{2.7}$$

$$a \to_5 b \overset{\text{def}}{=} ((a \cap b) \cup (a' \cap b)) \cup (a' \cap b') \tag{2.8}$$

quantum conjunctions as

$$a \cap_i b \overset{\text{def}}{=} (a \to_i b')', \qquad i = 1, \ldots, 5 \tag{2.9}$$

and quantum disjunctions as

$$a \cup_i b \stackrel{\text{def}}{=} a' \to_i b, \qquad i = 1, \ldots, 5 \tag{2.10}$$

Classical implication, disjunction, and implication are denoted as $a \to_0 b \stackrel{\text{def}}{=} a' \cup b$, $a \cup_0 b \stackrel{\text{def}}{=} a \cup b$, and $a \cap_0 b \stackrel{\text{def}}{=} a \cap b$, respectively.

For a subsequent use we introduce the following definition of an orthomodular lattice.

Definition 2.4. An ortholattice to which any of the following equivalent orthomodularity conditions are added:

$$a \to_i b = 1 \quad \Leftrightarrow \quad a \leqslant b, \qquad i = 1, \ldots, 5 \tag{2.11}$$

is called an orthomodular lattice, OML.[30]

For the quantum operations defined above we can prove the following lemmas:

Lemma 2.5. Quantum De Morgan law holds in any ortholattice for quantum conjunction and disjunction:

$$a \cap_i b = (a' \cup_i b')', \qquad i = 1, \ldots, 5 \tag{2.12}$$

Lemma 2.6. In any OML the following quantum ordering relations hold for quantum conjunctions and disjunctions:

$$a \leqslant_{\cup i+} b \stackrel{\text{def}}{\Leftrightarrow} a \cup_i b = b, \qquad i = 1, 3, 4, 5 \tag{2.13}$$

$$a \leqslant_{\cup i-} b \stackrel{\text{def}}{\Leftrightarrow} b \cup_i a = b, \qquad i = 2, 3, 4, 5 \tag{2.14}$$

$$a \leqslant_{\cap i+} b \stackrel{\text{def}}{\Leftrightarrow} a \cap_i b = a, \qquad i = 1, 3, 4, 5 \tag{2.15}$$

$$a \leqslant_{\cap i-} b \stackrel{\text{def}}{\Leftrightarrow} b \cap_i a = a, \qquad i = 2, 3, 4, 5 \tag{2.16}$$

Proof. One straightforwardly checks reflexivity, antisymmetry, and transitivity. In particular, if we use Lemma 2.7 below (whose proof does not depend on this lemma), the proof becomes trivial. Negative results (for $i = 1, 2$) follow from failures in OML MO2 (Fig. 1a). □

In what follows we sometimes use $\leqslant_i$ to denote any of $\leqslant_{\cup i+}$, $\leqslant_{\cup i-}$, $\leqslant_{\cap i+}$, $\leqslant_{\cap i-}$ from Lemma 2.6 (excluding the exceptions for $i = 1, 2$). One can easily prove the following two lemmas, using the fact that each condition fails in the non-orthomodular lattice O6 (Fig. 1b): ((17), p. 22)

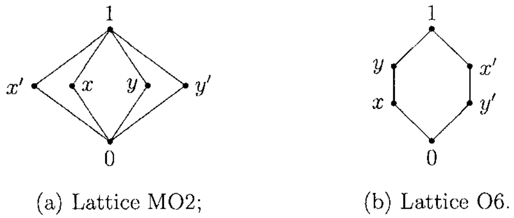

Fig. 1. (a) Lattice MO2; (b) lattice O6.

Lemma 2.7.

$$a \leqslant_i b \quad \Leftrightarrow \quad a \leqslant b, \qquad \text{for all } i \text{ from Lemma 2.6}$$

make OL orthomodular and hold in any OML.

Lemma 2.8. An ortholattice to which any of the following conditions is added:

$$a' \cup_i b = 1 \quad \Leftrightarrow \quad a \leqslant_{\cup i+} b, \qquad i = 1, 3, 4, 5 \tag{2.17}$$

$$a' \cup_i b = 1 \quad \Leftrightarrow \quad a \leqslant_{\cup i-} b, \qquad i = 2, 3, 4, 5 \tag{2.18}$$

$$a' \cup_i b = 1 \quad \Leftrightarrow \quad a \leqslant_{\cap i+} b, \qquad i = 1, 3, 4, 5 \tag{2.19}$$

$$a' \cup_i b = 1 \quad \Leftrightarrow \quad a \leqslant_{\cap i-} b, \qquad i = 2, 3, 4, 5 \tag{2.20}$$

is an OML and vice versa.

The main theorem of this section is:

Theorem 2.9. Conditions OL1–OL3 and OL5–OL9 as well the orthomodularity (Definition 2.4) hold with $\leqslant_i$ from Lemma 2.6 substituted for $\leqslant$ and with $\cup_i$ and $\cap_i$, $i = 1, \ldots, 5$ substituted, for $\cup$ and $\cap$, respectively. As for OL4 we have:

(OL4) $$a \leqslant_{\cap 1+} a \cup_1 b \;\&\; b \nleqslant_{\cap 1+} a \cup_1 b \;\&\; a \nleqslant_{\cap 2-} a \cup_2 b \;\&\; b \leqslant_{\cap 2-} a \cup_2 b \;\&\; a \nleqslant_i a \cup_i b \;\&\; b \nleqslant_i a \cup_i b \qquad i = 3, 4, 5$$

In addition, $a \cup_i a' = a' \cup_i a = 1$ and $a \cap_i a' = a' \cap_i a = 0$ hold for any i.

Proof. The positive results are easy to verify if we convert $\leqslant_i$ to $\leqslant$ using Lemma 2.7. For OL5, we illustrate the proof for $i = 3$ and $a \cup_3 b$:

from $a \leqslant c$ and $b \leqslant c$, we have $a \cap b \leqslant c$, $a \cap b' \leqslant c$, and $a' \cap (a \cup b) \leqslant c$. Hence $(a \cap b) \cup (a \cap b') \cup (a' \cap (a \cup b)) \leqslant c$. The negative results for OL4 are shown by their failures in OML MO2. □

Lemma 2.10. The following forms of a deduction theorem make any OL orthomodular. (We say that b is deducible from Γ if there exist $a_1, a_2, \ldots, a_n \in \Gamma$ such that $a_n \cap_1 (a_{n-1} \cap_1 (\ldots a_1)\ldots) \leqslant_{1+} b$, where $\leqslant_{1+}$ denotes any one of $\leqslant$, $\leqslant_{\cup 1+}$, $\leqslant_{\cap 1+}$.)[31]

$$(b \cap_1 a) \leqslant c \quad \Leftrightarrow \quad a \leqslant b \to_1 c \text{ [20, 22]} \tag{2.21}$$

$$(b \cap_1 a) \leqslant_{\cup 1+} c \quad \Leftrightarrow \quad a \leqslant_{\cup 1+} b \to_1 c \tag{2.22}$$

$$(b \cap_1 a) \leqslant_{\cap 1+} c \quad \Leftrightarrow \quad a \leqslant_{\cap 1+} b \to_1 c \tag{2.23}$$

and these conditions hold in any orthomodular lattice, OML. Equations (2.22) and (2.23) do not hold for any other i from Lemma 2.6 substituted together with the corresponding $b \cap_i a$ for $\leqslant_{1+}$ and $b \cap_1 a$, respectively.

Consequences of Theorem 2.9. If we keep only to *quantum operations* and *quantum ordering* we will arrive at an algebra which is not a lattice (in the sense that $\cup_1$ and $\cap_1$ operations do not coincide with supremum and infimum) because the lattice axiom OL4 is not satisfied. How this algebra can be axiomatized (starting with those above conditions that hold in a lattice) is an open problem. Specifically, such an axiomatization would be able to prove those conditions that hold in any OML when $\cup$ and $\cup_1$ are simultaneously interchanged throughout, i.e., conditions OL1-3, valid parts of OL4, OL5-9, and the orthomodularity. In any case the algebra cannot be made orthoisomorphic to the lattice of closed subspaces of Hilbert space. If we still wanted to consider an algebra based only on *quantum operations* and *quantum ordering* we would have the following alternatives: To start with valid conditions from Theorem 2.9, take them as axioms of an algebra, and add additional axioms so as to possibly eventually arrive at a finite axiomatization of the algebra. Or, to express standard operations from OML by means of quantum ones and arrive at algebras we will consider in the next section. Finite axiomatizability of such algebras is also an open question (except for those studied in Sec. 6). □

3. OPERATION ALGEBRAS $QA_{[i]}$

As follows from Theorem 2.9 we cannot have a proper "quantum" lattice based on "quantum operations," "quantum ordering," and equipped with a deduction theorem since we cannot have any lattice based on them. However, following the approach of Ref. 19 we can arrive at the following algebra.

Definition 3.1. Operation algebra $QA_{[i]}$ is an algebra $\langle \mathscr{A}_0, ', \Cup \rangle$ such that the following rule is satisfied.

Substitution Rule. Any valid condition or equation one can obtain in the standard formulation of OML containing only variables, $\cup_1$ (where i is one, or some, or all of $i = 0, \ldots, 5$), and negation written in $QA_{[i]}$ with $\Cup$ substituted for $\cup_i$ is a valid condition or equation in QA. Subscript $[i]$ denotes the range of i. (For brevity, we omit braces when i is an n-tuple, so $QA_{[0,1]}$ denotes $QA_{[\{0,1\}]}$.)

Below, by giving explicit lattice expressions for $i = 0$, $i = 1$, and $i =$ 0–5, we implicitly define algebras $QA_{[0]}$ (standard OML), $QA_{[1]}$ (with $a \Cup b = a \cup_1 (a' \cap_1 b)$), and $QA_{[0\text{–}5]}$ (with $a \Cup b = a \cup_i (b \cup_i (b' \cap_i (a \cup_i (a \cap_i b'))))$, $i = 0, \ldots, 5$).

Algebra $QA_{[1]}$ does contain a deduction theorem and one can embed orthomodular lattice in it, but it is not a lattice (in the sense of $\cup_1$ and $\cap_1$ coinciding with supremum and infimum) although it shows a striking similarity to a classical distributive lattice. The similarity is contained in the form of orthomodularity (Lemma 2.8) and in the deduction theorem (Lemma 2.10). However, if we agree that a deduction theorem is not a very useful theorem in a lattice theory then Theorem 2.9 shows that there is no particular reason to limit ourselves to $\cup_1$ for $\Cup$. For example, in algebra $QA_{[1-5]}$ one can express all relevant operations from an orthomodular lattice by means of 5 quantum disjunctions and conjunctions at once (e.g., $a \cup b = ((a \cup_i b') \cup_i (b' \cup_i a))' \cup_i a$, $i = 1, \ldots, 5$).[19] This might seem to take us away from any "classical" feature of the algebra, but below we show that in quantum-classical algebra $QA_{[0\text{–}5]}$ one can express all expressions in an orthomodular lattice by means of either quantum or classical conjunctions and disjunctions in structurally identical ways. More explicitly, we express 96 possible two-variable expressions in an orthomodular lattice (so-called Beran expressions[29]) by means of both quantum and classical disjunctions and conjunctions, and negation.

4. CLASSICAL-QUANTUM ALGEBRA $QA_{[0\text{–}5]}$

Let $n_{a,b}$, $n = 1, \ldots, 96$, $a, b \in \mathrm{OML}$ be a Beran expression.[29] As we stressed in Ref. 19 there are 16 classical and 80 quantum Beran expressions.

Classical expressions can be expressed by classical-quantum disjunctions and conjunctions (where $\Cup = \cup_i$, $i = 0, \ldots, 5$; DeMorgan's law holds $a \Cap b = (a' \Cup b')'$, but we use both operations for the compactness of expressions) and negation. Shortest such expressions are given below. We also give shortest expressions by means of $\cup_1$ and $\cap_1$.

$$1 = 0' = 96_{a,b} = 1'_{a,b} = 1 = 0'$$

$$a = 22_{a,b} = 75'_{a,b} = 39_{b,a} = 58'_{b,a} = a$$

$$a \cup b = (a' \cap b')' = 92_{a,b} = 93_{a,b'} = 94_{a',b} = 95_{a',b'} = 2'_{a,b} = 3'_{a,b'} = 4'_{a',b} = 5'_{a',b'}$$
$$= a \Cup (b \Cup (b' \Cap (a \Cup (a \Cap b')))) = b \cup_1 (b' \cap_1 a)$$

$$a \equiv_0 b = (b \cup a') \cap (b' \cup a) = 88_{a,b} = 9'_{a,b}$$
$$= (a \Cup b) \Cap ((a \Cap b)' \Cap ((a \Cap b') \Cup (a' \Cap b))) = (b' \cap_1 a') \cup_1 (b \cap_1 a)$$

Quantum Beran expressions are: *quantum unities*, 1_{abi}, *quantum zeros*, 0_{abi}, $i = 1, 2$, *quantum variables* a_{abi}, $i = 1, 2, 3$, and the above defined *quantum identities*, *disjunctions* and *conjunctions*, that all reduce to their classical counterparts for compatible variables. They can be expressed by classical-quantum disjunctions and conjunctions [DeMorgan's law holds $a \Cap b = (a' \Cup b')'$], and negation as given below (shortest possible expressions). We also give their shortest expressions by means of $\cup_1$ and $\cap_1$.

$$1_{ab1} = 0'_{a'b'1} = ((a \cap b) \cup (a \cap b')) \cup ((a' \cap b) \cup (a' \cap b')) = 16_{a,b} = 81'_{a,b}$$
$$= ((a \Cup b) \Cup (b \Cup a')) \Cap (((a' \Cap b) \Cup (b \Cap a)) \Cup ((a \Cup b)' \Cup (b' \Cap a)))$$
$$= (a \cup_1 (b \cup_1 a')) \cap_1 (a' \cup_1 (b \cup_1 a))$$

$$1_{ab2} = 0'_{a'b'2} = (a \cup (a' \cap b)) \cup (a' \cap b')$$
$$= 32_{a,b} = 80_{a',b'} = 48_{b,a} = 64_{b',a'} = 17'_{a',b'} = 65'_{a,b} = 33'_{b',a'} = 49'_{b,a}$$
$$= a \Cup (((b \Cup (a \Cup b)') \Cap (a \Cup (a \Cap b))') \Cup a) = a \cup_1 (b \cap_1 a)'$$

$$a_{b1} = ((a \cap b) \cup (a \cap b')) = 6_{a,b} = 7_{b,a} = 10_{b',a'} = 11_{a',b'}$$
$$= 86'_{a',b'} = 87'_{b',a'} = 90'_{b,a} = 91'_{a,b}$$
$$= a \Cap ((a \Cup b) \Cap (a' \Cup ((b \Cap a) \Cup (a \Cap b')))) = a \cap_1 (a' \cup_1 (b \cup_1 a))$$

$$a_{b2} = (a \cup b) \cap (b' \cup (b \cap a)) = 54_{a,b} = 23_{b',a} = 26'_{b',a'}$$
$$= 38_{a,b'} = 43'_{a',b'} = 59'_{a,b'} = 71_{b,a} = 74'_{b',a}$$
$$= (b' \Cup (a \Cup b')) \Cap ((a \Cap b) \Cup (((b \Cup a) \Cup a) \Cap b')) = (b' \cap_1 a) \cup_1 a$$

$$a_{b3} = ((a \cup b) \cap (a \cup b')) \cap ((a' \cup (a \cap b)) \cup (a \cap b'))$$

$$= 70_{a,\,b} = 27_{a',\,b'} = 55_{b,\,a} = 42_{b',\,a'}$$

$$= (a' \Cup (b \Cup a)) \Cap (((b \Cap a) \Cup (a \Cap b')) \Cup ((a \Cup ((b \Cup a) \Cap b')) \Cap a'))$$

$$= (a' \cap_1 (b' \cap_1 a)) \cup_1 ((b' \cap_1 a) \cup_1 a)$$

$$a \equiv_1 b = a' \equiv_3 b' = (a \cup b') \cap (a' \cup (a \cap b)) = 72_{a,\,b} = 73_{a,\,b'} = 56_{b,\,a} = 57_{b,\,a'}$$

$$= (a \Cup (b \Cup a))' \Cup ((b' \Cup a) \Cap ((b \Cap (a \Cap b)) \Cup a'))$$

$$= (a \cup_1 b)' \cup_1 (b \cap_1 a)$$

$$a \equiv_2 b = a' \equiv_4 b' = (a \cup b') \cap (b \cup (b' \cap a')) = 40_{a,\,b} = 41_{a',\,b} = 24_{b,\,a} = 25_{a,\,b'}$$

$$= (b \Cap (a \Cap b)) \Cup ((a \Cup b') \Cap (b \Cup ((b \Cup a)' \Cup b)))$$

$$= (b \cap_1 a) \cup_1 (a \cup_1 (b \cup_1 a))'$$

$$a \equiv_5 b = (b' \cap a') \cup (b \cap a) = 8_{a,\,b} = 89'_{a,\,b}$$

$$= (a \Cup b') \Cap ((a \Cup (b \Cup a))' \Cup ((b \Cap a) \Cap b)) = (b \cup_1 a') \cap_1 (b' \cup_1 a)$$

$$a \cup_1 b = b \cup_2 a = (a' \cap_1 b')' = (b' \cap_2 a')' = a \cup (a' \cap b)$$

$$= 28_{a,\,b} = 29_{a,\,b'} = 44_{b,\,a} = 46_{b,\,a'} = 61_{b',\,a} = 63_{b',\,a'} = 78_{a',\,b} = 79_{a',\,b'}$$

$$= 18'_{a',\,b'} = 19'_{a',\,b} = 34'_{b',\,a'} = 36'_{b',\,a} = 51'_{b,\,a'} = 53'_{b,\,a} = 68'_{a,\,b'} = 69'_{a,\,b}$$

$$= (a \Cup (b \Cap (a \Cup (a \Cap b))'))$$

$$a \cup_3 b = b \cup_4 a = (a' \cap_3 b')' = (b' \cap_4 a')'$$

$$= (a \cup b) \cap ((a' \cup (a \cap b')) \cup (a \cap b))$$

$$= 76_{a,\,b} = 30_{a',\,b} = 31_{a',\,b'} = 45_{b',\,a} = 47_{b',\,a'} = 60_{b,\,a} = 62_{b,\,a'} = 77_{a,\,b'}$$

$$= 66'_{a',\,b'} = 20'_{a,\,b'} = 21'_{a,\,b} = 35'_{b',\,a} = 37'_{b,\,a} = 50'_{b',\,a'} = 52'_{b',\,a} = 67'_{a',\,b}$$

$$= (a' \Cup (b \Cup a)) \Cap (((b \Cap a) \Cup (a \Cap b')) \Cup ((a \Cup b) \Cap a'))$$

$$= (a' \cap_1 b) \cup_1 (b \cup_1 a)$$

$$a \cup_5 b = (a' \cap_5 b')' = ((a \cap b) \cup (a \cap b')) \cup (a' \cap b)$$

$$= 12_{a,\,b} = 13_{a',\,b} = 14_{a',\,b'} = 15_{a,\,b'} = 82'_{a',\,b'} = 83'_{a',\,b} = 84'_{a,\,b} = 85'_{a,\,b'}$$

$$= (a \Cup b) \Cap (((a' \Cap b) \Cap (b \Cup a)) \Cup ((b \Cap a) \Cup (a \Cap b')))$$

$$= (a \cup_1 b) \cap_1 (a' \cup_1 (b' \cup_1 a))$$

Each of the 96 $\Cup$, $\Cap$ expressions above (which are polynomials in $\Cup$, $\Cap$, $'$, 0, 1) is a shortest representative of an (infinite) equivalence class of polynomials of $QA_{[0-5]}$ that evaluate (in an OML) to the same Beran expression for each of $\cup_i$, $\cap_i$, $i = 0,\ldots, 5$ substituted for $\Cup$, $\Cap$. In addition to these polynomials, there are many other polynomials that do not evaluate to the same Beran expression for all i, for example $a \Cup b$. Every polynomial belongs to an equivalence class of polynomials that can be connected with the $=$ sign. For example, $a \cup_i b = a \cup_i (b \cup_i (a \cap_i (a \cup_i b)))$ holds for $i = 0,\ldots, 5$, so $a \Cup (b \Cup (a \Cap (a \Cup b)))$ belongs to the same equivalence class as $a \Cup b$. In fact, there are $2^4 \times 6^6 = 746496$ such equivalence classes of expressions with at most 2 distinct variables.

By means of an exhaustive computer search, the authors determined that a shortest polynomial for a member of each equivalence class requires from 0 to 14 variable occurrences. We have 0 and 1 with no variable occurrences; a, b, a', b' with one occurrence; $a \Cup b$, $b \Cup a$, $a \Cap b'$, etc. (16 cases) with two occurrences; and so on. An example of a shortest representative of a class requiring at least 14 variable occurrences is $(a \Cap (b \Cap (b \Cup a))) \Cup ((a' \Cap ((b \Cup a) \Cap b)) \Cup ((b \Cup ((b \Cup (a \Cup b)) \Cap a')) \Cap b'))$. The number of equivalence classes vs. the number of variable occurrences required for a shortest representative is shown in Table I. The computer search yielded a list, available from the authors, with a shortest representative of each of the 746496 classes. (The list also includes as subsets shortest representatives for all possible operation algebras $QA_{[i]}$, $i \subseteq \{0,\ldots, 5\}$. The next section shows how these can be extracted from the list.)

There are usually several shortest representatives of a given equivalence class, and the relationship is not always obvious. For example, $a \Cap (b \Cup a)' = a \Cap (b \Cap a')$ are two shortest representatives of one class. Other interesting classes include those that can permute quantum operations to others; for example $((b' \Cap a) \Cap (a \Cup b)) \Cup ((a \Cap b) \Cup (b \Cap a'))$, i.e., $((b' \cap_i a) \cap_i (a \cup_i b)) \cup_i ((a \cap_i b) \cup_i (b \cap_i a'))$ evaluates to $a \cup_{5-i} b$, $i = 0,\ldots, 5$.

The situation for $\Cup$, $\Cap$ expressions with 3 or more distinct variables appears to be much more complicated and is poorly understood. Apparently

Table I. (a) Number of Equivalence Classes for $QA_{[0-5]}$ Polynomials with at Most 2 Distinct Variables (Total 746469); (b) Number of Variable Occurrences in a Shortest Representative of those Classes

(a)	2	4	16	224	1926	10568	29444	101168	195380	204296	138584	48852	14272	1684	76
(b)	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14

the only known non-trivial conditions with 3 variables in $QA_{[0\text{–}5]}$ are the Foulis–Holland-like associativity and distributivity properties presented in Ref. 24, the condition OL5 of Theorem 2.9, and the following conditions for the Sasaki projection $\varphi_a(b) \overset{\text{def}}{=} a \cap (a' \cup b) = a \Cap (b \Cup (a \Cap (a \Cup b))')$.

Theorem 4.1. The following conditions hold in $QA_{[0\text{–}5]}$:

$$\varphi_a(c) = \varphi_b(c) \quad \Rightarrow \quad a \Cap c = b \Cap c \tag{4.1}$$

$$\varphi_a(c') = \varphi_b(c') \quad \Rightarrow \quad a \Cup c = b \Cup c \tag{4.2}$$

Proof. For (4.1), we successively operate on both sides of the hypothesis with either c or a previous equality, using the identities $a \cap c = c \cap \varphi_a(c)$, $a \cap_1 c = \varphi_a(c)$, $a \cap_2 c = (c \to_1 (\varphi_a(c))')'$, $a \cap_3 c = (((a \cap_1 c) \to_1 c) \to_1 (a \cap_2 c)')'$, $a \cap_4 c = ((c \to_1 (a \cap_1 c)) \to_1 (a \cap_1 c)')'$, and $a \cap_5 c = (a \cap_1 c) \cup (a \cap_2 c)$ on the left-hand side (and the analogous ones for b on the right-hand side). For (4.2), we use the identity $\varphi_{a'}(c') = (c' \to_2 \varphi_a(c'))'$ after operating on both sides with c, then use (4.1) to obtain $a' \Cap c' = b' \Cap c'$, from which the result follows by complementing both sides and applying DeMorgan's law. □

Comment From the identity $\varphi_{a'}(c) = (c \to_2 \varphi_a(c))'$, we also note the interesting consequence $\varphi_a(c) = \varphi_b(c) \Leftrightarrow \varphi_{a'}(c) = \varphi_{b'}(c)$, which holds in all OMLs and is equivalent to the orthomodular law. This is also equivalent to $a \to_1 c = b \to_1 c \Leftrightarrow a' \to_1 c = b' \to_1 c$.

An open problem is whether $QA_{[0\text{–}5]}$ can be represented with a finite set of equations. □

5. CONSTRUCTION OF $QA_{[0\text{–}5]}$ EXPRESSIONS

In the previous section we showed several examples of shortest representatives of the 746496 equivalence classes for 2- variable $QA_{[0\text{–}5]}$ expressions, which were found by exhaustive search. In this section we describe an algorithm that will produce a representative of any desired equivalence class. Although the expressions produced with this algorithm will usually be extremely long and not practical to work with, the algorithm is nonetheless instructive as it illustrates another way to describe and classify the equivalence classes. It also provides a proof that all possible 746496 equivalence classes can be represented in $QA_{[0\text{–}5]}$.

The 96 Beran expressions correspond to the 96 elements of the free OML $F(a, b)$ with 2 free generators a, b. Each element can be separated into a "Boolean part" and an "MO2 part.[32]" A Beran expression can be identified with an ordered pair of numbers, the first of which (0 through 15) identifies the Boolean part n_{B} and the second (0 through 5) the MO2 part n_{M}. We choose these numbers in such a way that Beran's numbering n_{Beran} (1 through 96) is recovered by $n_{\mathrm{Beran}} = 16n_{\mathrm{B}} + n_{\mathrm{M}} + 1$. Conversely, given the Beran number the Boolean part is $n_{\mathrm{B}} = (n_{\mathrm{Beran}} - 1) \bmod 16$ and the MO2 part is $n_{\mathrm{M}} = \lfloor (n_{\mathrm{Beran}} - 1)/16 \rfloor$.

The Boolean part of any expression of $\mathrm{QA}_{[0\text{–}5]}$ is the same for $i = 0,\ldots, 5$ when $\cup_i$, $\cap_i$ are substituted for $\Cup$, $\Cap$. The MO2 part can differ for each i. We represent any equivalence class by a septuple $\langle n_{\mathrm{B}}, n_{\mathrm{M}_0},\ldots, n_{\mathrm{M}_5}\rangle$ where n_{B} identifies the Boolean part and $n_{\mathrm{M}_0},\ldots, n_{\mathrm{M}_5}$ the MO2 parts.

When $n_{\mathrm{M}_0} = \cdots = n_{\mathrm{M}_5}$, the expression is in the equivalence class of one of the 96 Beran expressions. For example, $a \cup b = a \Cup (b \Cup (b' \Cap (a \Cup (a \Cap b'))))$ from the previous section, with Beran number 92, is in the equivalence class $\langle 11, 5, 5, 5, 5, 5, 5\rangle$. When the n_{M_i} are not all the sane, the equivalence class does not correspond to any Beran expression. For example, $a \Cup b$ is in the equivalence class $\langle 11, 5, 1, 2, 4, 3, 0\rangle$. The example $((b' \Cap a) \Cap (a \Cup b)) \Cup ((a \Cap b) \Cup (b \Cap a'))$ from the previous section is in the equivalence class $\langle 11, 0, 3, 4, 2, 1, 5\rangle$, allowing one to easily see that it evaluates to $a \cup_{5-i} b$, $i = 0,\ldots, 5$ by reversing the order of the six MO2 components. Continuing with some other examples from the previous section, we have $\langle 0, 0, 0, 0, 0, 0, 0\rangle$ for 0, $\langle 15, 5, 5, 5, 5, 5, 5\rangle$ for 1, $\langle 5, 1, 1, 1, 1, 1, 1\rangle$ for a, $\langle 6, 2, 2, 2, 2, 2, 2\rangle$ for b, $\langle 10, 4, 4, 4, 4, 4, 4\rangle$ for a', $\langle 9, 3, 3, 3, 3, 3, 3\rangle$ for b', $\langle 1, 0, 1, 2, 4, 3, 5\rangle$ for $a \Cap b$, $\langle 1, 0, 2, 1, 3, 4, 5\rangle$ for $b \Cap a$, $\langle 0, 0, 1, 0, 4, 1, 1\rangle$ for $a \Cap (b \Cap a')$, and $\langle 6, 3, 5, 5, 0, 3, 3\rangle$ for $(a \Cap (b \Cap (b \Cup a))) \Cup ((a' \Cap ((b \Cup a) \Cap b)) \Cup ((b \Cup ((b \Cup (a \Cup b)) \Cap a')) \Cap b'))$.

For operation algebras other than $\mathrm{QA}_{[0\text{–}5]}$, a representative expression can be obtained by simply ignoring the omitted MO2 component(s). For example, to obtain a shortest representative for $a \cup b$ in $\mathrm{QA}_{[1]}$, we look at each septuple of the form $\langle 11, ., 5, ., ., ., .\rangle$ where "." means "don't care," and pick a shortest from the list of 746496 mentioned in the previous section. In this case a shortest is $a \cup b = a \cup_1 (a' \cap_1 b)$, obtained from $a \Cup (a' \Cap b)$ in equivalence class $\langle 11, 1, 5, 2, 1, 2, 5\rangle$. There are also 3 other shortest ones: $b \Cup (b' \Cap a)$ in $\langle 11, 2, 5, 1, 2, 1, 5\rangle$, $(a' \Cap b) \Cup a$, in $\langle 11, 1, 5, 1, 1, 4, 5\rangle$, and $(b' \Cap a) \Cup b$ in $\langle 11, 2, 5, 2, 2, 3, 5\rangle$.

Now we are ready to describe an algorithm for constructing a representative expression for a given septuple $\langle n_{\mathrm{B}}, n_{\mathrm{M}_0},\ldots, n_{\mathrm{M}_5}\rangle$. In what follows, for brevity we will interchangeably use expressions and their corresponding septuples. Our construction starts with the Boolean part chosen from

Table II. Equivalence Classes and Shortest Representatives for the Boolean Part of a $QA_{[0-5]}$ Construction. These Correspond to Beran Expressions 81 Through 96, Respectively

$\langle 0, 5, 5, 5, 5, 5, 5\rangle$	$((a \Cap b) \Cap (b \Cap a')) \Cup (((a' \Cup b) \Cap (b \Cup a)) \Cap ((a \Cap b)' \Cap (b' \Cup a)))$
$\langle 1, 5, 5, 5, 5, 5, 5\rangle$	$(a \Cap b) \Cup (((a' \Cup b) \Cap (b \Cup a)) \Cap ((b \Cup a) \Cap (a \Cup b')))$
$\langle 2, 5, 5, 5, 5, 5, 5\rangle$	$(a \Cap b') \Cup (((a \Cap b)' \Cap (b \Cup a)) \Cap ((b' \Cup a) \Cap (a \Cup b)))$
$\langle 3, 5, 5, 5, 5, 5, 5\rangle$	$(b \Cap a') \Cup (((b \Cap a)' \Cap (a \Cup b)) \Cap ((a' \Cup b) \Cap (b \Cup a)))$
$\langle 4, 5, 5, 5, 5, 5, 5\rangle$	$((a \Cup b) \Cap (((a' \Cap b) \Cap (b \Cup a)) \Cup ((b \Cap a) \Cup (a \Cap b'))))'$
$\langle 5, 5, 5, 5, 5, 5, 5\rangle$	$a \Cup ((a \Cap b) \Cup (a' \Cap ((b \Cup a) \Cap (a \Cup b'))))$
$\langle 6, 5, 5, 5, 5, 5, 5\rangle$	$b \Cup ((a \Cap b) \Cup (a' \Cap ((b \Cup a) \Cap (a \Cup b'))))$
$\langle 7, 5, 5, 5, 5, 5, 5\rangle$	$(a \Cap b) \Cup ((a \Cup b)' \Cup ((a \Cup b') \Cap (a' \Cup b)))$
$\langle 8, 5, 5, 5, 5, 5, 5\rangle$	$(a \Cap b') \Cup ((a \Cap (b \Cap a))' \Cap ((b \Cup a) \Cup b))$
$\langle 9, 5, 5, 5, 5, 5, 5\rangle$	$b' \Cup ((a \Cup b)' \Cup (a \Cap ((b \Cap a)' \Cap (a \Cup b))))$
$\langle 10, 5, 5, 5, 5, 5, 5\rangle$	$a' \Cup ((a \Cup b)' \Cup (((a \Cup b) \Cap (b \Cap a)') \Cap b))$
$\langle 11, 5, 5, 5, 5, 5, 5\rangle$	$a \Cup (b \Cup (b' \Cap (a \Cup (a \Cap b'))))$
$\langle 12, 5, 5, 5, 5, 5, 5\rangle$	$a \Cup (b' \Cup (b \Cap (a \Cup (a \Cap b))))$
$\langle 13, 5, 5, 5, 5, 5, 5\rangle$	$b \Cup (a' \Cup (a \Cap (b \Cup (b \Cap a))))$
$\langle 14, 5, 5, 5, 5, 5, 5\rangle$	$(a \Cap (b' \Cap a)) \Cup (a \Cap (b \Cap a))'$
$\langle 15, 5, 5, 5, 5, 5, 5\rangle$	1

Table II, obtaining $\langle n_B, 5, 5, 5, 5, 5, 5\rangle$. We then operate on each of the MO2 components in succession, changing them from 5 to the desired value.

For the MO2 components that we want to keep unchanged, we make use of the fact that $a \Cap 1 = a$ for any a. More specifically, the expression $\langle n_B, \ldots, n_{M_i}, \ldots\rangle \Cap \langle 15, \ldots, 5, \ldots\rangle$ is equal to some other expression $\langle n_B, \ldots, n_{M_i}, \ldots\rangle$ where MO2 component n_{M_i} is unchanged but the other MO2 components are possibly different.

The expressions from Table III allow us to operate on specific MO2 components in succession while keeping all other MO2 components unchanged. (The table omits the cases where $n_{M_i} = 5$ since in those cases there is nothing to be done.) To construct the MO2 part for $i = 0$, we pick $\langle 15, n_{M_0}, 5, 5, 5, 5, 5\rangle$ from the first part of Table III and obtain

$$\langle n_B, 5, 5, 5, 5, 5, 5\rangle \Cap \langle 15, n_{M_0}, 5, 5, 5, 5, 5\rangle = \langle n_B, n_{M_0}, 5, 5, 5, 5, 5\rangle \quad (5.1)$$

To construct the MO2 part for $i = 1$, we pick $\langle 15, 5, n_{M_1}, 5, 5, 5, 5\rangle$ from the second part of Table III and obtain

$$\langle n_B, n_{M_0}, 5, 5, 5, 5, 5\rangle \Cap \langle 15, 5, n_{M_1}, 5, 5, 5, 5\rangle = \langle n_B, n_{M_0}, n_{M_1}, 5, 5, 5, 5\rangle \quad (5.2)$$

Continuing in this fashion for $i = 2, 3, 4, 5$, we finally arrive at an expression representing $\langle n_B, n_{M_0}, \ldots, n_{M_5}\rangle$.

Table III. Equivalence Classes and Shortest Representatives for the MO2 Parts of a $QA_{[0–5]}$ Construction. These Have no Corresponding Beran Expressions

⟨15, 0, 5, 5, 5, 5, 5⟩	$(a \Cap b) \Cup ((a \Cap b') \Cup (a' \Cap (a' \Cap ((b \Cup a)' \Cup b))))$
⟨15, 1, 5, 5, 5, 5, 5⟩	$a \Cup ((a' \Cap b) \Cup (b \Cup a)')$
⟨15, 2, 5, 5, 5, 5, 5⟩	$b \Cup ((b' \Cap a) \Cup (a \Cup b)')$
⟨15, 3, 5, 5, 5, 5, 5⟩	$b' \Cup ((b \Cap a) \Cup (a' \Cap b))$
⟨15, 4, 5, 5, 5, 5, 5⟩	$a' \Cup ((a \Cap b) \Cup (b' \Cap a))$
⟨15, 5, 0, 5, 5, 5, 5⟩	$(a \Cap (a' \Cup b)) \Cup ((a \Cup (b \Cup a')) \Cap ((a \Cap b)' \Cup b))$
⟨15, 5, 1, 5, 5, 5, 5⟩	$a \Cup (a \Cup ((b \Cup a) \Cup a'))$
⟨15, 5, 2, 5, 5, 5, 5⟩	$b \Cup (a \Cup (b \Cap (b \Cup a))')$
⟨15, 5, 3, 5, 5, 5, 5⟩	$b' \Cup (a \Cup ((b \Cap a) \Cup b))$
⟨15, 5, 4, 5, 5, 5, 5⟩	$a' \Cup (b \Cup ((a \Cap b) \Cup a))$
⟨15, 5, 5, 0, 5, 5, 5⟩	$((a \Cup b) \Cap b') \Cup ((a \Cup (a' \Cup b)) \Cap ((b \Cup a) \Cup b'))$
⟨15, 5, 5, 1, 5, 5, 5⟩	$a \Cup (b \Cup (b' \Cup (b \Cup a)))$
⟨15, 5, 5, 2, 5, 5, 5⟩	$a \Cup (a' \Cup ((a \Cup b) \Cap b))$
⟨15, 5, 5, 3, 5, 5, 5⟩	$a \Cup (a \Cap ((a \Cup b) \Cap b))'$
⟨15, 5, 5, 4, 5, 5, 5⟩	$b \Cup (b \Cap ((b \Cup a) \Cap a))'$
⟨15, 5, 5, 5, 0, 5, 5⟩	$(a \Cap (a \Cup (b \Cup a))) \Cup (a \Cup (a \Cap (b \Cup a)))'$
⟨15, 5, 5, 5, 1, 5, 5⟩	$a \Cup (a' \Cup (b \Cup a))$
⟨15, 5, 5, 5, 2, 5, 5⟩	$b \Cup (b' \Cup (a \Cup b))$
⟨15, 5, 5, 5, 3, 5, 5⟩	$b \Cup (a \Cup (a \Cup b)')$
⟨15, 5, 5, 5, 4, 5, 5⟩	$a \Cup (b \Cup (b \Cup a)')$
⟨15, 5, 5, 5, 5, 0, 5⟩	$(a \Cap ((a \Cup b) \Cup a)) \Cup (a \Cup ((a \Cup b) \Cap a))'$
⟨15, 5, 5, 5, 5, 1, 5⟩	$a \Cup ((a \Cup b) \Cup a')$
⟨15, 5, 5, 5, 5, 2, 5⟩	$b \Cup ((b \Cup a) \Cup b')$
⟨15, 5, 5, 5, 5, 3, 5⟩	$a \Cup ((a \Cup b)' \Cup b)$
⟨15, 5, 5, 5, 5, 4, 5⟩	$b \Cup ((b \Cup a)' \Cup a)$
⟨15, 5, 5, 5, 5, 5, 0⟩	$(a \Cup b) \Cup (a \Cap b)'$
⟨15, 5, 5, 5, 5, 5, 1⟩	$a \Cup (a \Cup (b \Cup (b \Cup (b \Cap a)')))$
⟨15, 5, 5, 5, 5, 5, 2⟩	$b \Cup (a \Cup ((a \Cap b)' \Cup b))$
⟨15, 5, 5, 5, 5, 5, 3⟩	$b' \Cup (b \Cup (a \Cup (a \Cup b)))$
⟨15, 5, 5, 5, 5, 5, 4⟩	$a' \Cup (a \Cup (b \Cup (b \Cup a)))$

6. FINITE AXIOMATIZATIONS FOR $QA_{[0]}, \ldots, QA_{[5]}$

As we mentioned above, it is an open problem whether in general $QA_{[i]}$ is finitely axiomatizable in general for $i \subseteq \{0, \ldots, 5\}$. However, when i is a singleton, finite axiomatizations exist. $QA_{[0]}$ is of course just standard OML. We show them for each singleton in what follows by displaying a specific finite axiomatization.

For the following definition we stress that $\cup$ is a *defined* (not primitive) operation.

Definition 6.1. An algebra $\mathrm{QA}'_{[i]}$ (where i is one of 0,..., 5) is a triple $\langle \mathscr{A}_i, ', \Cup \rangle$ satisfying axioms $\mathrm{QA}1'_i$–$\mathrm{QA}10'_i$, where $\mathrm{QA}1'_i$–$\mathrm{QA}8'_i$ are displayed identically to axioms OL1–OL8 of Definition 2.1 and for any $a, b \in \mathscr{A}_i$:

$$\mathrm{QA}9'_i \qquad a \leqslant b \Rightarrow a \cup (a' \cap b) = b$$

$$\mathrm{QA}10'_i \qquad a \Cup b = a \cup_i b$$

where

$$a \Cap b \overset{\text{def}}{=} (a' \Cup b')'$$

$$a \cup b \overset{\text{def}}{=} a \Cup (b \Cup (b' \Cap (a \Cup (a \Cap b'))))$$

$$a \cap b \overset{\text{def}}{=} (a' \cup b')'$$

$$a \leqslant b \overset{\text{def}}{\Leftrightarrow} a \cup b = b$$

and where $\cup_i$ is defined as in Definition 2.3 only with the above defined $\cup$ and $\cap$ which are themselves defined by means of the primitive $\Cup$.

Theorem 6.2. Algebra $\mathrm{QA}'_{[i]}$ is an axiomatization for (i.e., is equivalent to) algebra $\mathrm{QA}_{[i]}$ from Definition 3.1 where i is one of 0,..., 5.

Proof. It is straightforward to show that each of $\mathrm{QA}1'_i$–$\mathrm{QA}10'_i$ holds in an OML when $\Cup$ is replaced with $\cup_i$ and thus is an axiom of $\mathrm{QA}_{[i]}$ according to the Substitution Rule.

For the converse, we note that the axioms $\mathrm{QA}1'_i$– $\mathrm{QA}9'_i$ are *structurally* identical to the axioms for an OML (compare Definition 2.1 to $\mathrm{QA}1'_i$–$\mathrm{QA}8'_i$, using the definition of $\cap$ for the structure of OL9, and adding $\mathrm{QA}9'_i$ for the structure of the OML law). With this structure, we can mimic an OML proof of any condition (equation or inference) involving (in addition to $'$) only the defined symbol $\cup$.

Suppose E is an axiom of $\mathrm{QA}_{[i]}$ per the Substitution Rule. We write down a condition E' with only $\cup$ symbols, obtained from E by expanding all $\Cup$ symbols according to the right-hand-side of $\mathrm{QA}10'_i$. Then E' will be structurally identical to a condition that holds in OML (because $\mathrm{QA}10'_i$ is structurally identical to the OML definition for $\cup_i$), so E' can be proved using $\mathrm{QA}1'_i$–$\mathrm{QA}9'_i$ to mimic the OML proof. Then from E' we obtain E by applying $\mathrm{QA}10'_i$. □

The axioms for $\mathrm{QA}'_{[i]}$ can be quite long when expressed in terms of the primitive $\Cup$. Some economy can be achieved by replacing the definition of

$\cup$ with 6 different ones, one for each i, by making use of the OML identities $a \cup b = a \cup_0 b = a \cup_1 (a' \cap_1 b) = b \cup_2 (b' \cap_2 a) = a \cup_3 (a \cup_3 b) = a \cup_4 (b \cup_4 a) = a \cup_5 (a' \cap_5 b)$. In addition, $QA'_{[0]}$ reduces to standard OML, and for $QA'_{[1]}$ an implicational algebra such as the one in Ref. 33 might provide a starting point for a more compact system.

For singleton i, it is easy to see that each $QA_{[i]}$ is equivalent to an OML system as the next theorem shows.

Theorem 6.3. Algebras $QA'_{[i]}$, where i is one of 0,..., 5, are equivalent to an OML (and thus to each other and to each $QA_{[i]}$ as well).

Proof. For one direction, we define $\cup$ in terms of $\Cup$ per Definition 6.1. For the other direction, we define $\Cup$ in terms of $\cup$ by treating $QA10'_i$ as a definition. That the axioms of each system are satisfied in the other is a straightforward verification. □

For algebras such as $QA_{[0-5]}$ with non-singleton i, the equivalence in the above sense does not hold. For, although we can define $a \cup b$ in terms of $a \Cup b$, we cannot define $a \Cup b$ in terms of $a \cup b$. With the first definition, we can write down axioms from the Substitution Rule that correspond to the axioms for an OML, but there are additional axioms such as $a \Cap (b \Cup a)' = a \Cap (b \Cap a')$ that have no corresponding $\cup$ version under this embedding.

7. CONCLUSION

Our results have shown that the so-called *quantum operations* cannot be used for building an operation algebra underlying Hilbert space. In particular, Theorem 2.9 proves that no quantum operation can satisfy all lattice conditions. Hence, the aforementioned operation algebra cannot have the lattice of closed subspaces of Hilbert space as its model. This is not a problem though, because in Hilbert lattice the standard conjunction $a \cap b$, corresponds to set intersection, $\mathcal{H}_a \bigcap \mathcal{H}_b$, of subspaces $\mathcal{H}_a$, $\mathcal{H}_b$ of Hilbert space $\mathcal{H}$, the ordering relation $a \leqslant b$ corresponds to $\mathcal{H}_a \subseteq \mathcal{H}_b$, the standard disjunction $a \cup b$, corresponds to the smallest closed subspace of $\mathcal{H}$ containing $\mathcal{H}_a \bigcup \mathcal{H}_b$, and a' corresponds to $\mathcal{H}_a^{\perp}$, the set of vectors orthogonal to all vectors in $\mathcal{H}_a$. We then prove that Hilbert lattice is orthoisomorphic to the lattice of closed subspaces of any Hilbert space and this is what we need Hilbert lattice for. From the framework of quantum theory and/or Hilbert space theory no need for new algebras based on *quantum operations* emerges.

The whole issue of quantum operations have recently been put forward by a reconsideration of the so-called *deduction theorem* within a lattice theory originally formulated by Finch 30 years ago.[20] Román and Zuazua[22] attempted to give a new formal and Hilbert space interpretation of Finch's formulation. Essentially they advocated the usage of the quantum operations the theorem is based on, instead of the standard lattice operations. Other authors also recently advocated their usage as "especially interesting from a physical point of view.[24]" However, they all failed to substitute quantum for standard operations in the definition of the ordering relation too, and this is what we did in Sec. 2. Our Theorem 2.9 shows that a consistent substitution of quantum for standard operations take us to algebras none of which is a lattice and for which we do not know whether they are all finitely axiomatizable. To be more specific, when we use only quantum operations—even to define the ordering relation—then it turns out that the set of equations we get, satisfies all but one conditions of an orthomodular lattice. Therefore, we can try to use the conditions which are satisfied by quantum operations according to Theorem 2.9 to eventually arrive at a hypothetical well defined algebraic system which might even be finitely axiomatizable. Or, we can disregard these conditions altogether and simply express all standard operations by means of quantum ones. This is what we did in Sec. 3. We arrive at well defined algebras which we call *operation algebras* in analogy to implication algebras.[33, 28] Some of them are finitely axiomatizable ("singleton" operation algebras from Sec. 6) and for the others this is still an open question (see Sec. 4). All operation algebras properly contain any orthomodular lattice and "singleton" operation algebras are moreover completely equivalent to OML and to each other as we show in Sec. 6. Still, none of these algebras *is* a lattice (in the sense of satisfying condition OL4 of Theorem 2.9) and therefore they can hardly play any role in the quantum theory from a foundational point of view.

However, it is not a problem but a virtue of quantum logic that the only way to formulate it as an orthomodular lattice is exactly that one which maps the simplest operations defined on elements of Hilbert space. Once we have a lattice structure it is actually completely irrelevant which operation we use and this is what we have shown in Sec. 3: All 96 Beran expressions of an orthomodular lattice (quantum logic) can be formulated by "merged" conjunction and disjunction in such a way that we can substitute either classical or any of the five quantum conjunctions or disjunctions for the "merged" ones at will. For example in $a \cup_3 b = ((a' \Cup (b \Cup a)) \Cap (((b \Cap a) \Cup (a \Cap b')) \Cup ((a \Cup b) \Cap a')))$ we can substitute either $\cup$ and $\cap$, or $\cup_1$ and $\cap_1$, or $\cup_5$ and $\cap_5$, etc., for $\Cup$ and $\Cap$, respectively, and we will always get, $a \cup_3 b$ on the left hand side. Hence, it is not "logical properties" of operations but lattice structure what is essential. This lattice structure of Hilbert space we will most probably soon use as a basis for the architecture of a would-be quantum computer in a similar way we use the Boolean algebra for a classical computer. Here, an equational Hilbert lattice theory partially or even completely equivalent to the Hilbert space theory would be an appropriate tool for constructing finite dimensional quantum logic for quantum computers. Of such equations, one class of state-determined orthomodular lattice equations of nth order has been known for 20 years.[6] The second such class of Hilbert operator determined orthomodular lattice equations—orthoarguesian equations of nth order—was found only recently.[34, 9] Very recently we found a third such class—possibly the last one. All these equations are structurally determined by the properties of Hilbert space. No semantic considerations whatsoever enter any of the algorithms which served for finding the equations.

ACKNOWLEDGMENT

One of us (M. P.) would like to acknowledge support of the Ministry of Science of Croatia through the Project No. 082006.

REFERENCES

1. B. Christianson, P. L. Knight, and T. Beth, Implementations of quantum logic, *Philos. Trans. Roy. Soc. London Ser. A* **356**(1743), 1823–1838 (1998).
2. G. W. Mackey, *The Mathematical Foundations of Quantum Mechanics* (Benjamin, New York, 1963).
3. V. S. Varadarajan, *Geometry of Quantum Theory, Vols. 1 and 2* (Wiley, New York, 1968/1970).
4. H. A. Keller, Ein nicht-klassischer Hilbertscher Raum, *Math. Z.* **172**, 41–49 (1980).
5. B. M. Boghosian and W. Taylor, Simulating quantum mechanics on a quantum computer, *Phys. D* **120**, 30–42 (1998); http://xxx.lanl.gov/abs/quant-ph/9701019.
6. R. Godowski, Varieties of orthomodular lattices with a strongly full set of state, *Demonstratio Math.* **14**, 725–733 (1981).
7. R. Godowski and R. Greechie, Some equations related to the states on orthomodular lattices, *Demonstratio Math.* **17**, 241–250 (1984).
8. R. Mayet, Equational bases for some varieties of orthomodular lattices related to states, *Algebra Universalis* **23**, 167–195 (1986).
9. N. D. Megill and M. Pavičić, Equations, states, and lattices of infinite-dimensional Hilbert space, *Internat. J. Theoret. Phys.* **39**, 2349–2391 (2000); http://xxx.lanl.gov/abs/quant-ph/0009038.
10. M. L. Dalla Chiara, Quantum logic, in *Hand-book of Philosophical Logic*, *Vol. III*, D. Gabbay and F. Guentliner, eds. (Reidel, Dordrecht, 1986), pp. 427–469.
11. M. L. Dalla Chiara and R. Giuntini, Quantum logic, http://xxx.lanl.gov/abs/quant-ph/0101028 (2001).

12. P. F. Gibbins, *Particles and Paradoxes: The Limits of Quantum Logics* (Cambridge University Press, Cambridge, 1987).
13. M. Jammer, *The Philosophy of Quantum Mechanics. The Interpretations of Quantum Mechanics in Historical Perspective* (Wiley, New York, 1974).
14. G. Kalmbach, *Orthomodular Lattices* (Academic, London, 1983).
15. P. Mittelstaedt, *Quantum Logic* (Synthese Library, Vol. 18) (Reidel, London, 1978).
16. C. Piron, *Foundations of Quantum Physics* (Benjamin, Reading, Massachusetts, 1976).
17. G. T. Rüttimann, *Logikkalkúle der Quantenphysik. Eine Abhandlung zur Ermittlung der formal-logischen Systeme, die der nicht-relativistischen Quantentheorie zugrundeliegen* (Duncker & Humblot, Berlin, 1977).
18. M. Pavičić and N. D. Megill, "Non-orthomodular models for both standard quantum logic and standard classical logic: Repercussions for quantum computers," *Helv. Phys. Acta* **72**, 189–210 (1999); http://xxx.lanl.gov/abs/quart-ph/9906101.
19. N. D. Megill and M. Pavičić, "Orthomodular lattices and a quantum algebra," *Internat. J. Theoret. Phys.* **40**, 1387–1410 (2001); http://xxx.lanl.gov/abs/quant-ph/0103135.
20. P. D. Finch, Quantum logic as an implication algebra, *Bull. Austral. Math. Soc.* **2**, 101–106 (1970).
21. L. Román and B. Rumbos, Quantum logic revisited, *Found. Phys.* **21**, 793–734 (1991).
22. L. Román and R. E. Zuazua, Quantum implication, *Internat. J. Theoret. Phys.* **38**, 793–797 (1999).
23. J. Malinowski, The deduction theorem for quantum logic—some negative results, *J. Symbolic Logic* **55**, 615–625, 1990.
24. B. D'Hooghe and J. Pykacz, On some new operations on orthomodular lattices, *Internat. J. Theoret. Phys.* **39**, 641–652 (2000).
25. H. Dishkant, The first-order predicate calculus based on the minimal logic of quantum mechanics, *Rep. Math. Logic* **3**, 9–18 (1974).
26. G. N. Georgacarakos, Equationally definable implication algebras for orthomodular lattice, *Stud. Logica* **39**, 5–18 (1980).
27. G. M. Hardegree, Material implication in orthomodular (and Boolean) lattice, *Notre Dame J. Formal Logic* **22**, 163–182 (1981).
28. M. Pavičić and N. D. Megill, Quantum and classical implication algebras with primitive implications, *Internat. J. Theoret. Phys.* **37**, 2091–2098 (1998).
29. L. Beran, *Orthomodular Lattices; Algebraic Approach* (Reidel, Dordrecht, 1985).
30. M. Pavičić, Minimal quantum logic with merged implications, *Internat. J. Theoret. Phys.* **26**, 845–852 (1987).
31. R. I. Goldblatt, Semantic analysis of orthologic, *J. Philos. Logic* **3**, 19–35 (1974).
32. M. Navara, On generating finite orthomodular sublattices, *Tatra Mt. Math. Publ.* **10**, 109–117 (1997).
33. J. C. Abbott, Orthoimplication algebra, *Stud. Logica*, **35**, 173–177 (1976).
34. B. D. McKay, N. D. Megill, and M. Pavičić, Algorithms for Greechie diagrams, *Internat. J. Theoret. Phys.* **39**, 2393–2417 (2000); httl)://xxx.lanl.gov/abs/quant-ph/0009039.