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## New Kochen–Specker sets in four dimensions

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### ABSTRACT

We show that all possible 388 4-dim Kochen–Specker (KS) (vector) sets (of yes–no questions) with 18 through 23 vectors and 844 sets with 24 vectors all with component values from  $\{-1, 0, 1\}$  can be obtained by stripping vectors off a single system provided by Peres 20 years ago. In addition to them, we have found a number of other KS sets with 22 through 24 vectors. We present the algorithms we used and features we found, such as, for instance, that Peres' 24-24 KS set has altogether six critical KS subsets.

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### 1. Introduction

The Kochen–Specker (KS) theorem has recently been given renewed attention due to new theoretical results which then prompted new experimental and computational techniques.

The new theoretical results concern the conditions under which such experiments are feasible at all [1–4], including a possible way to formulate the KS theorem for single qubits [5,6]. Such results and experiments enable applications in quantum computation (restrictions imposed on complex configurations of quantum gates, implementations of KS configurations of quantum gates that rule out classical solutions, etc.).

The experiments were carried out for spin- $\frac{1}{2} \otimes \frac{1}{2}$  particles (correlated photons or spatial and spin neutron degrees of freedom), and therefore in this Letter we provide results only for 4-dim KS vector sets of yes–no questions (KS sets for short). The first experiments and their designs [7–11] were not literal KS sets. They made use of state-dependent vector orientations that were additionally “translated” into new measurable observables according to ingenious keys found by their authors, because they could not be directly implemented by reading off the orientations of the vectors from the original set. The most recent designs and experiments [12–17] dispense with state-dependent vectors.

The configurations of Hilbert space vectors and subspaces in KS sets have interesting symmetries that have intrigued many authors since the very discovery of the KS theorem. The set of Kochen

and Specker themselves [18] is a highly symmetrical structure that consists of three identical substructures each of which consists of five hexagons. This perfect symmetry enabled Kochen and Specker to “see” and prove their theorem. Other highly symmetrical 3-dimensional constructions were given by Peres [19] and Penrose [20]. Both constructions can be given a straightforward physical interpretation (by means of either rays in a 3-dim spin-1 Hilbert space—equivalent to those in Euclidean space—or by means of Majorana spin representation) and an appealing geometrical visualisation on a cube [21] or on Escher's Waterfall ornament [20].

Similar symmetries were found for higher spins, i.e., higher dimensions. In four dimensions, even more symmetries have been found. Peres has found a highly symmetrical 24 system of 24 vectors grouped in 24 tetrads each consisting of 4 mutually orthogonal vectors [19]. This system can be seen as a geometrical representation of Mermin's elegant set for a pair of two spin- $\frac{1}{2}$  subsystems ( $\frac{1}{2} \otimes \frac{1}{2}$ ) [22] which has recently been experimentally realised [14]. Another representation, based on a dodecahedron (consisting of 12 pentagons and containing 40 rays), has been given by Penrose [20,23,24]. Its physical representation is based on a Majorana representation of a pair of entangled spin- $\frac{3}{2}$  systems. In these examples, a geometric visualisation is not as direct as in the aforementioned 3-dim cases, where we can make use of a Euclidean space instead of a Hilbert space. Nevertheless it can help us find appropriate experimental sets even in cases with a much higher number of rays, corresponding to the number of measurements and preparations of a system or the number of gates we pass the system through—depending on the kind of experiment.

For instance, KS sets recently considered by Aravind and Lee-Elkin are based on the geometry of two 4-dimensional polytopes,

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the 600-cell (each cell congruent to a regular tetrahedron) and the 120-cell, which provide us with highly symmetrical configurations of 300 and 60 KS rays, respectively [25]. These highly symmetrical structures (usually a particular regular group) can be extended to higher dimensions [26], but they also contain many KS sub-structures. For example, the smallest 4-dim 18-9 system found by Cabello et al. [27], a 20-11 found by [28], and a number of systems with 19 through 24 rays and 10 through 24 tetrads found by Pavičić et al. [29] were all found to be contained in Peres' 24-24 system [30].

These findings motivated us to find out how many possible KS sets there are, how many of them are sub-sets of larger ones, and whether we can generate them from each other. This is, however, a rather complex task which cannot be carried out as a straightforward counting, simply because all today's clusters and grids together would take many ages of the Universe to carry it out using a brute force approach. The complexity of the direct approach is illustrated, e.g., by the fact that it took seven years before Gould and Aravind [31] succeeded in comparing just two of such systems—the aforementioned Peres' and Penrose's 3-dim KS systems—and proving them isomorphic to each other. Instead, we developed a new way of describing and visualising KS systems, wrote many new algorithms and programs, and discovered many new symmetries and other features of the systems.

In Ref. [29] we gave algorithms for exhaustive generation of KS system containing arbitrary number of vectors with all possible numbers of their blocks in any number of dimensions with vector component values from any set. Then we scanned all the systems with up to (but not including) 23 vectors in a 4-dim space. In the meantime, we also scanned all the systems with 23 vectors, and the results revealed many new features. In particular, it turned out that all possible KS vector sets with up to and including 23 vectors and with components from the set  $\{-1, 0, 1\}$  are contained in Peres' 24-24 set. It was obvious that by simply peeling off blocks of vectors from the 24-24 set we get many more subsets. But then we discovered that by using a lattice representation of vector rays and filtering them through non-dispersive states that we can define on these lattices, we get exactly the KS subsets contained in the 24-24 set. Via this method, we get additional 844 24-vector systems contained in the 24-24 set. We conjecture that these are all possible KS sets with 24 vectors with components from the set  $\{-1, 0, 1\}$ .

We also found 37 new KS sets with 22 through 24 vectors with component values from other sets (not from  $\{-1, 0, 1\}$ ).

## 2. Algorithms

To obtain our results we used the algorithms that are described in detail in [29] and some others that we describe in Appendix A.

We start by describing vectors as vertices (points) and orthogonalities between them as edges (lines connecting vertices), thus obtaining MMP diagrams [32,30,33] which are defined as follows:

1. Every vertex belongs to at least one edge;
2. Every edge contains at least 3 vertices;
3. Edges that intersect each other in  $n - 2$  vertices contain at least  $n$  vertices.

We denote vertices of MMP diagrams by  $1, 2, \dots, A, B, \dots, a, b, \dots$ . There is no upper limit for the number of vertices and/or edges in our algorithms and/or programs.

Isomorphism-free generation of MMP diagrams follows the general principles established by [34], which we now recount briefly. Deleting an edge from an MMP diagram, together with any vertices that lie only on that edge, yields another MMP diagram (perhaps the vacuous one with no vertices). Consequently, every MMP diagram can be constructed by starting with the vacuous diagram and

adding one edge at a time, at each stage obtaining a new MMP diagram. We can represent this process as a rooted tree whose vertices correspond to MMP diagrams, in which the vertices and edges have unique labels. The vacuous diagram is at the root of the tree, and for any other diagram its parent node is the diagram formed by deleting the edge with the highest label. The isomorph rejection problem is to prune this tree until it contains just one representative of each isomorphism class of diagram.

To find diagrams that cannot be ascribed 0–1 values, we apply an algorithm which we call `states01` and which is based on the lattice theory of Hilbert space states. The algorithm is an exhaustive search of MMP diagrams with backtracking. The criterion for assigning 0–1 (dispersion-free) states is that each edge must contain exactly one vertex assigned to 1, with the others assigned to 0. As soon as a vertex on an edge is assigned a 1, all other vertices on that edge become constrained to 0, and so on.

## 3. Results

To find KS vectors, we follow the idea put forward in [32,30] and proceed so as to require that their number, i.e. the number of vertices within edges, corresponds to the dimension of the experimental space  $\mathbb{R}^n$  and that edges correspond to  $n(n - 1)/2$  equations resulting from inner products of vectors being equal to zero (meaning orthogonality). So, e.g., an edge of length 4, BCDE, represents the following 6 equations:

$$\begin{aligned} \mathbf{a}_B \cdot \mathbf{a}_C &= a_{B1}a_{C1} + a_{B2}a_{C2} + a_{B3}a_{C3} + a_{B4}a_{C4} = 0, \\ \mathbf{a}_B \cdot \mathbf{a}_D &= a_{B1}a_{D1} + a_{B2}a_{D2} + a_{B3}a_{D3} + a_{B4}a_{D4} = 0, \\ \mathbf{a}_B \cdot \mathbf{a}_E &= a_{B1}a_{E1} + a_{B2}a_{E2} + a_{B3}a_{E3} + a_{B4}a_{E4} = 0, \\ \mathbf{a}_C \cdot \mathbf{a}_D &= a_{C1}a_{D1} + a_{C2}a_{D2} + a_{C3}a_{D3} + a_{C4}a_{D4} = 0, \\ \mathbf{a}_C \cdot \mathbf{a}_E &= a_{C1}a_{E1} + a_{C2}a_{E2} + a_{C3}a_{E3} + a_{C4}a_{E4} = 0, \\ \mathbf{a}_D \cdot \mathbf{a}_E &= a_{D1}a_{E1} + a_{D2}a_{E2} + a_{D3}a_{E3} + a_{D4}a_{E4} = 0. \end{aligned} \quad (1)$$

Each possible combination of edges for a chosen number of vertices corresponds to a system of such nonlinear equations. A solution to systems which correspond to MMP diagrams without 0–1 states is a set of components of KS vectors we want to find. Thus the main method for finding *all* KS vectors is to exhaustively generate all MMP diagrams, then pick out all those diagrams that cannot have 0–1 states, then establish the correspondence between the latter diagrams and the equations for the vectors as shown in Eq. (1), and finally solve the systems of the so obtained equations.

To find solutions in the set  $\{-1, 0, 1\}$  we use the program `vectorfind`, and to find solutions in the set of real numbers we use the interval analysis as described in detail in [29,35]. There is no other upper limit for the number of vertices and edges of the generated MMP diagrams and solved equations apart from the computational power of today's supercomputers.

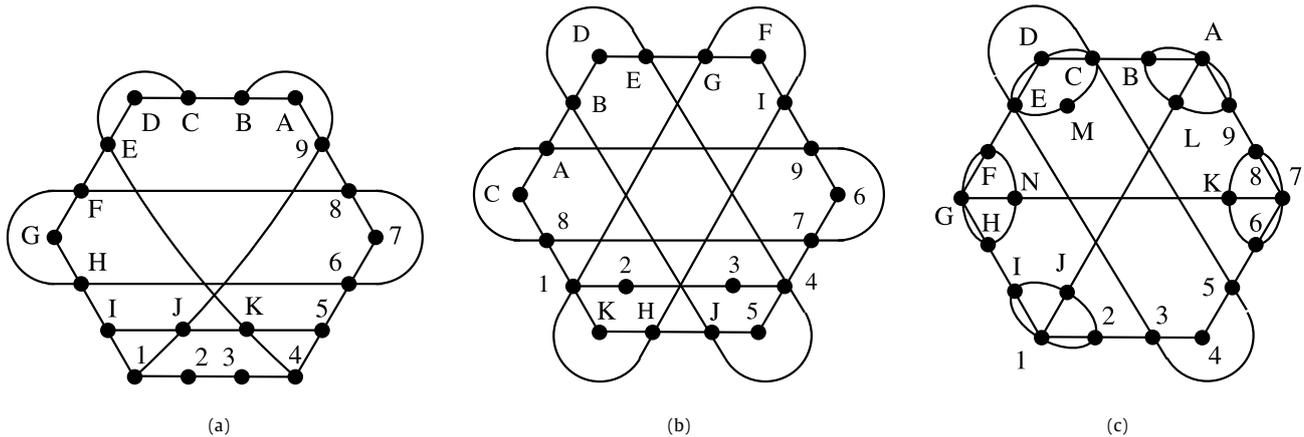
In Table 1 we give the numbers of all 18- through 24-vector sets with component values from  $\{-1, 0, 1\}$  that we generated and solved with the help of the aforementioned algorithms and programs. Vector sets with vector component values from other sets than  $\{-1, 0, 1\}$  are given at the end of the Letter.

We reported on the properties of the KS sets with 18 through (including) 22 vectors in [29].<sup>1</sup> It took two weeks on our cluster with 500 3.4 GHz processors (recalculated) in 2004. For the present results, we ran a parallel computation for 23-vector sets and obtained the 275 sets given in the 6th row of Table 1. This

<sup>1</sup> Notice that here (as opposed to [29]) the sets with loops of size 2 and 3 are put together.

**Table 1**  
KS sets for systems with 4 degrees of freedom with up to 24 vectors with component values from  $\{-1, 0, 1\}$ .

\	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	Total
18	1																1
19		1															1
20		1	5	1													7
21			2	11	4	1											18
22			1	9	36	23	12	3	1								85
23				2	19	76	79	58	27	11	3	1					276
24				1	6	39	137	187	188	136	83	41	18	6	2	1	845
Total	1	2	8	24	65	139	228	248	216	147	86	42	18	6	2	1	1233



**Fig. 1.** KS sets: (a) 20-10; (b) the same 20-10 but redrawn so as to match the visual appearance of 18-9 in Fig. 1 from [36] and Fig. 3 of [29]; (c) 23-14 which contains neither 18-19 nor (a), (b), (c) from Fig. 3 of [29].

took about two months on our cluster. We then analysed the data and conjectured that all sets with solutions from  $\{-1, 0, 1\}$  might be subsets of the aforementioned 24-24 set. Our program subgraph confirmed the conjecture.

That meant that we can actually get all 18- through 24-vector sets by stripping vectors and tetrads—vertices and edges in MMP notation—of the 24-24 set and filtering it with our state01 program described in [29]. We wrote the program subset to generate all subsets (i.e. MMP diagrams with edges removed) of the 24-24 set. From these, we determined the ones with 18 through 23 vectors that are isomorphic with the ones we previously obtained on our cluster. It is interesting that all such stripped sets filtered by state01 have solutions. In addition, we determined (again filtering the output of subset) 844 24-vector sets with 12 through 23 tetrads (MMP diagrams with 24 vertices with 12 through 23 edges). They are given in the seventh row of Table 1. All that, i.e., obtaining all 1232 sets shown in Table 1 with their vector component values from the 24-24-set, took a few minutes on a single PC.

In that way we can even get new sets with up to 41 vectors (upper limit for the solutions from  $\{-1, 0, 1\}$  [29]) simply by adding new vectors and tetrads to the sets from the 7th row of Table 1.

For a higher number of vertices we might find KS sets that do not contain any of the sets from Table 1 as their subsets. If their vectors had their component values from the set  $\{-1, 0, 1\}$ , they should have loops of order higher than six because they should not have any of the above 1231 sets as their subsets. With today's computer power, such a search is not feasible, though.

We analysed the obtained vector sets and obtained the properties we present below. All the vector sets contain a hexagon MMP loop 1234, 4567, 789A, ABCD, DEFG, GHI1 which is always given in our figures [except in Fig. 1(b)] and for which we assume it is present whenever we give a new KS set. For in-

stance, for 20-10 from Fig. 1(a) we just write: H68F, IJK5, 1J9B, 4KEC.

The set 20-10 contains the smallest system 18-9. To determine the orientations of its vectors, we use the program vectorfind. It gives the component values given in Table 2 of Appendix B.

Previously, we found two smallest (20-11) KS sets that do not contain the smallest 18-9 set (Fig. 4(a) and (b) of [29]) and two smallest (22-13) sets that contain neither of the previous sets (Fig. 4(c) and (d) of [29]).

Our new results show that there are two 23-14s that contain neither the above 18-9, nor the two 20-11s, nor the first of the above 22-13s. One of them, 12JI, 1JLA, 35CE, 678K, 9ABL, CDEM, FGHN, GNK7, is given in Fig. 1(c). It contains (d) from Fig. 3 of [29].

There are also two 23-14s that contain neither the above 18-9, nor the two 20-11s, nor the second of the above 22-13s. One of them, 12JI, 1J9B, 345K, 4KEC, 6LMB, 9ABM, FGHN, GNL7, is given in Fig. 2(a). It contains (c) from Fig. 3 of [29].

The vectors component values for the two KS sets are given in Table 2 of Appendix B.

In Fig. 2(b) we give the only set (24-15) that does not contain any of the previous sets. The set (c) is the one which contains all the previous sets. Their MMP notations can easily be read off their figures.

The vector component values for the two 23-14 KS sets are given in Table 2 of Appendix B.

Additional KS sets are given in Appendix B.

KS sets with vectors having component values from sets other than  $\{-1, 0, 1\}$  are less numerous than the ones with values from  $\{-1, 0, 1\}$ . They are not our primary target in this Letter and we shall present only several examples below while the exhaustive generation of these sets is under way [37].

All 37 KS sets with 22 through 24 vectors with component values from sets other than  $\{-1, 0, 1\}$  would have component values

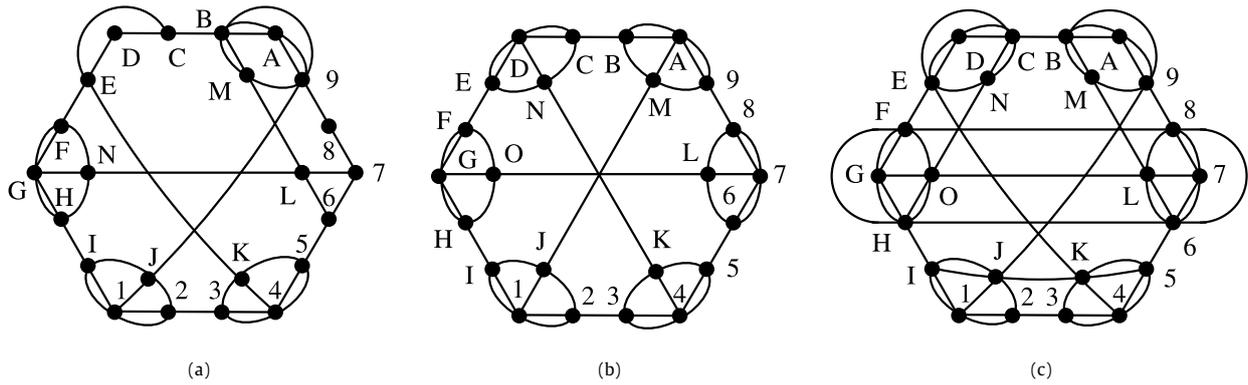


Fig. 2. (a) 23-14 which contains neither 18-19 nor (a), (b), (d) from Fig. 3 of [29]; (b) 24-15 set (the only one that exists) that does not contain any of the previous sets; (c) 24-20 that contains all previous sets.

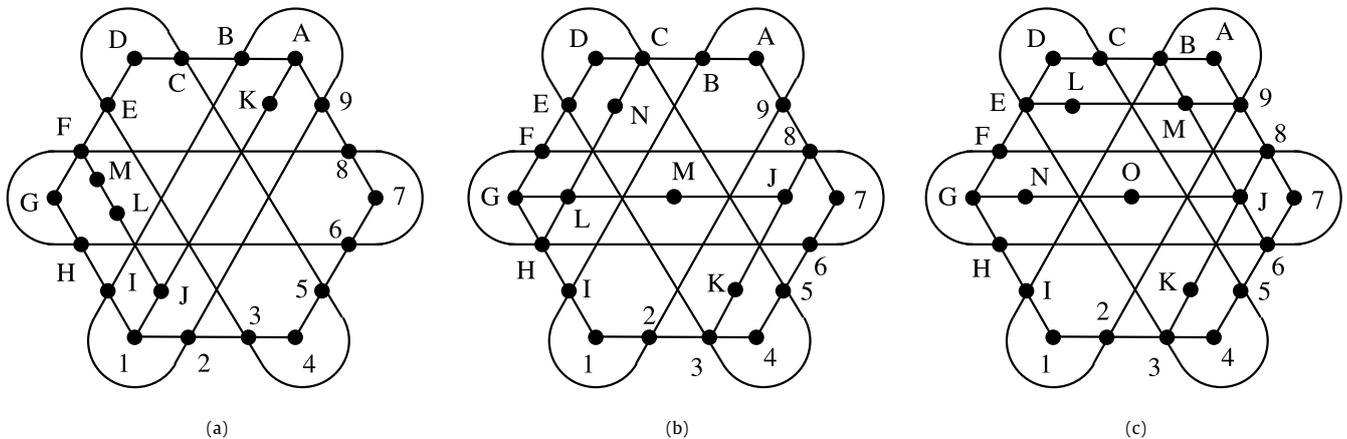


Fig. 3. KS sets with vectors whose components are not from  $\{-1, 0, 1\}$ : (a) 22-11; (b) 23-12; (c) 24-14.

from  $\{-1, 0, 1\}$  if we discarded vectors that share only one tetrad. But we clearly cannot do so because we have to have all vectors in every tetrad to be able to assign 1 to one and 0 to three of them. This confirms the results obtained in [29,38].

All of these sets contain the 18-9 set. The smallest one is 22-11: 25BE, 1AJK, JFLM, 68FH, 39IC shown in Fig. 3(a) It contains 20-10 from Fig. 1(a).

The vector component values for this KS set are given in Table 2 of Appendix B.

23-12 KS set shown in Fig. 3(b) contains the 18-9, the 20-10, and the 22-11. And 24-14 set shown in Fig. 3(c) contains the 18-9, the 20-10, a 21-11, the 22-11, a 22-12, a 22-13, and a 23-12. Additional such KS sets and vector components the reader can find in [37].

#### 4. Conclusions

We sum up our results as follows. All possible 388 KS sets for systems with 4 degrees of freedom with 18 through 23 vectors and 844 KS sets with 24 vectors with component values from  $\{-1, 0, 1\}$  can be obtained by “peeling” vectors off a single system provided—in effect—by Peres 20 years ago. But we would not know that the sets with 18 through 23 vectors obtained by such peeling exhaust all possible KS sets up to 23 vectors without extensive computation we carried out. And the computation would not have been feasible without putting together the theory of hypergraphs, lattice theory, and interval analysis, and many algorithms and programs we devised for the purpose.

Among particular features of KS sets we presented in Section 3, we would like to single out the one about the so-called *critical sets*, i.e., those KS sets that do not properly contain any KS subset

[23,25]. We found out that there are altogether six critical subsets of Peres’ 24-24 set. These are 18-9 [27], 20-11 [28], another 20-11 and two 22-13s [29], and 24-15 [given in Fig. 2(b)].

There exist sets with 22 and more vectors with component values that are not from  $\{-1, 0, 1\}$  and that are not isomorphic to any of the 1,233 sets mentioned above. Unlike the “ $\{-1,0,1\}$  sets,” they can be obtained only by extensive generation of MMP diagrams and computation of their properties, which we are currently carrying out [37].

As a final note, we mention that all 4-dim KS sets we have considered contain a single hexagon as the biggest loop formed by their tetrads. A geometrical interpretation of this fact is an open question, because in general there is no particular limit on the loop size in non-KS orthogonal tetrads of rays contained in 4-dim Hilbert space sets.

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#### Appendix A. Programs

The primary programs used for this project (and referenced earlier in this Letter) are *subgraph*, *states01*, *vectorfind*, and

subset.<sup>2</sup> Each is a stand-alone ANSI C program. Each program typically has several input, output, and other options, which can be listed with the `--help` option by typing e.g. `subgraph --help` at the Unix or Linux command-line prompt. Below we describe their main algorithms.

### A.1. *subgraph*

The program `subgraph` takes as its input two hypergraphs in the form of MMP diagrams, a test graph and a reference graph. It will indicate whether or not the test graph is a subgraph of the reference graph, using the following algorithm (suggested by Brendan McKay).

Let the edges of the test graph be  $w_1, w_2, \dots, w_m$  and of the reference graph  $v_1, v_2, \dots, v_n$ , where  $m \leq n$ . (If  $m > n$ , the subgraph relation will fail.) The problem is to find a sequence  $v'_1, \dots, v'_m \subseteq \{v_1, v_2, \dots, v_n\}$  such that the mapping  $w_i \mapsto v'_i$  ( $i = 1, \dots, m$ ) is an isomorphism. We construct this sequence one element at a time. For  $v'_1$ , we choose an element from  $\{v_1, \dots, v_n\}$ . For  $v'_2$ , we choose an element from  $\{v_1, \dots, v_n\}$  whose relationship (described below) to  $v'_1$  is the same as the relationship of  $w_2$  to  $w_1$ . If there is no such  $v'_2$ , we backtrack and choose another  $v'_1$ . Next,  $v'_3$  is anything whose relationship to  $v'_1, v'_2$  is the same as the relationship of  $w_3$  to  $w_1, w_2$ . And so on, backtracking recursively if necessary until a  $v'_m$  is found. If this process is successful, the subgraph relation holds; if, on the other hand, the backtracking is exhausted, the subgraph relation fails.

We determine the condition “ $v'_{k+1}$  is in the same relationship to  $v'_1, \dots, v'_k$  as  $w_{k+1}$  is to  $w_1, \dots, w_k$ ” as follows. Suppose we have chosen  $v'_1, \dots, v'_k$  and wish to find the possibilities for  $v'_{k+1}$ . We look at  $w_{k+1}$ : it has (e.g. for a 4-dim hypergraph) 4 vertices, and each is a member of some (possibly none) of the edges  $w_1, \dots, w[k]$ . So  $w_{k+1}$  gives a set of 4 subsets (in terms of indices) of  $\{1, 2, \dots, k\}$ . Similarly, any edge  $x$  in the reference graph gives a set of 4 subsets of  $\{1, 2, \dots, k\}$  that say which of the edges  $v'_1, \dots, v'_k$  contain each of the 4 vertices of  $x$ . The choices for  $v'_{k+1}$  are those edges of the reference graph which give the same 4 subsets of  $\{1, 2, \dots, k\}$  as  $w_{k+1}$  gives (and of course  $v'_{k+1}$  is different from  $v'_1, \dots, v'_k$ ).

### A.2. *states01*

The program `states01` indicates whether or not an MMP diagram can be assigned a non-dispersive state, in other words whether there exists a 0-1 assignment to all vertices such that each edge contains exactly one vertex assigned with 1. The program does an exhaustive search of all possible assignments to the MMP diagram, using a fast backtracking algorithm. More details are described in Ref. [29].

### A.3. *vectorfind*

The program `vectorfind` takes as its input an MMP diagram supplied by the user. It attempts to assign to each vertex a 3-dim or 4-dim vector (when the MMP diagram has 3 or 4 vertices per edge respectively), such that the following constraints are satisfied: (1) each vector, chosen from a predetermined set specified by the user, must be unique (non-parallel to all the others), and (2) the vectors assigned to the vertices in a given edge must be mutually orthogonal (i.e. have inner product equal to zero). If an assignment is found, it is printed out; otherwise the failure to find

one is indicated. The algorithm is an exhaustive search of all possible assignments from the used-specified vector set, using recursive backtracking.

A goal of the algorithm is to achieve extremely fast run time (compared to the more general interval analysis method described in Section 3). While worst-case run time can grow exponentially with the number of vertices, typically the program's speed is much faster. An internal optimisation processes vertices with the most edges before others to encourage early backtracking in the recursive search. A user-settable timeout will abandon the relatively rare attempts that take too long (and likely don't have a solution). For the standard KS sets in the literature, vector assignments are found almost instantaneously on a desktop computer.

### A.4. *subset*

The program `subset` is a relatively simple utility program that generates all subsets of the set of edges of its input MMP diagram. By default, subsets containing isolated edges (ones not connected to any other edge) are suppressed. The output will consist of  $2^{n-1}$  MMP diagrams minus the suppressed ones, where  $n$  is the number of edges. The program does not check for isomorphisms, so it is possible that some of its output diagrams are isomorphic to each other. (The program `subgraph` is one way to filter these if desired.)

## Appendix B. Additional results

Using our program `subgraph` and several ad hoc Linux scripts for collecting and filtering outputs we obtained the following results. The complete encoding of the KS sets we use below include hexagons which we only assumed in our figure representations above.

KL MN, GHIJ, DEFJ, BC FI, 9ABC, 78DE, 56GH, 1234, 34AC, 248E, 146H, 9CMN, 7ELN, 5HLM and KL MN, HIJN, DEFG, 9ABC, 5678, 234J, 178I, 1BCH, 4FGN, 68EG, ACDG, 23LM, 358M, 39CL are the other two 23-14 sets that do not contain Fig. 3(c) and Fig. 3(d) of [29], respectively and that do contain Fig. 3(d) and Fig. 3(c), respectively [we call them (c) and (d) below].

Other KS sets that contain neither 18-9, nor the two 20-11s are the following 23-15 (1), 24-14 (1), 24-15 (10), 24-16 (5), and 24-17 (2).

23-15 contains both (c) and (d):

KL MN, GHIJ, CDEF, ABEF, 9BIJ, 789A, DF MN, HJLN, 3456, 2568, 1347, 1278, 46CF, 45GJ, 18KN.

24-14 contains both (c) and (d):

LMNO, HIJK, DEFG, 9ABC, 5678, 1234, 78KO, BCJO, 34IN, FGHN, 2468, 14AC, 58EG, 9CDG.

24-15(2,4,6-9) do not contain (c) and 24-15-(1,3-5,10) do not contain (d); thus 24-15-(4) is a critical KS set isomorphic to the one given in Fig. 2(b):

LMNO, HIJK, DEFG, 9ABC, 5678, 1234, 3478, 24BC, 14FG, 68JK, ACIK, EGIJ, 58NO, 9CMO, DGMN.

LMNO, HIJK, DEFG, 9ABC, 5678, 1234, 34BC, 78AC, 24FG, 68EG, 9CJK, DG IK, 14NO, 58MO, IJMN.

LMNO, HIJK, DEFG, 9ABC, 5678, 1234, 34CK, 248O, 14FG, ABIJ, 67MN, 5BJO, 79KN, BEGI, 7DGM.

LMNO, HIJK, DEFG, 9ABC, 5678, 1234, 34FG, 78EG, BCDG, 24JK, 68IK, ACHK, 14NO, 58MO, 9CLO.

LMNO, HIJK, DEFG, 9ABC, 5678, 1234, 478K, 3BCK, 68FG, ACEG, 12IJ, 58NO, 9CMO, 2DGJ, 2IMN.

LMNO, HIJK, DEFG, 9ABC, 5678, 1234, 478O, 3BCO, 68FG, ACEG, 58JK, 9CIK, 12MN, 2DGN, 1HKN.

LMNO, HIJK, DEFG, 9ABC, 5678, 1234, 48KO, 7FGK, 3BCO, ACEG, 56IJ, 12MN, 69CJ, 2DGN, 26IM.

<sup>2</sup> They can be downloaded from <http://us.metamath.org/#ql> or from <http://m3k.grad.hr/ql>.

**Table 2**  
Table of vector component values for some Chosen KS sets.

	18-9	23-14a	23-14b	24-15	24-20	22-11
1	{1, 0, 0, 1}	{0, 0, 0, 1}	{0, 0, 0, 1}	{0, 0, 0, 1}	{0, 0, 0, 1}	{0, 1, 0, 0}
2	{0, 1, 0, 0}	{0, 0, 1, 0}	{1, 0, 0, 0}	{1, 0, 0, 0}	{0, 1, 1, 0}	{0, 0, 1, 0}
3	{0, 0, 1, 0}	{1, -1, 0, 0}	{0, 1, 1, 0}	{0, 1, 1, 0}	{1, 0, 0, 0}	{1, 0, 0, 0}
4	{1, 0, 0, -1}	{1, 1, 0, 0}	{0, 1, -1, 0}	{0, 1, -1, 0}	{0, 1, -1, 0}	{0, 0, 0, 1}
5	{1, 0, 1, 0}	{0, 0, 1, 1}	{1, 0, 0, -1}	{1, 1, 1, -1}	{1, 1, 1, 1}	$\{\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0, 0\}$
6	{1, -1, -1, 1}	{1, -1, 1, -1}	{1, 1, 1, 1}	{1, 1, 1, 1}	{1, 0, 0, -1}	$\{\frac{1}{2}, -\frac{1}{2}, -\frac{1}{\sqrt{2}}, 0\}$
7	{1, -1, -1, -1}	{1, -1, -1, 1}	{1, -1, -1, 1}	{1, -1, -1, 1}	{1, -1, -1, 1}	$\{\frac{1}{2}, -\frac{1}{2}, \frac{1}{\sqrt{2}}, 0\}$
8	{1, 1, 0, 0}	{1, 1, 1, 1}	{1, -1, 1, -1}	{1, 1, -1, -1}	{1, 0, 1, 0}	$\{\frac{1}{2}, \frac{1}{2}, 0, -\frac{1}{\sqrt{2}}\}$
9	{1, -1, 0, 0}	{1, 0, 0, -1}	{1, 1, 0, 0}	{0, 1, 0, 1}	{1, 1, -1, -1}	$\{0, \frac{1}{\sqrt{2}}, \frac{1}{2}, \frac{1}{2}\}$
A	{1, 1, 1, 1}	{0, 1, -1, 0}	{0, 0, 1, 1}	{1, 0, 1, 0}	{0, 1, 0, 1}	$\{\frac{1}{\sqrt{2}}, 0, -\frac{1}{2}, \frac{1}{2}\}$
B	{1, 1, -1, -1}	{1, 0, 0, 1}	{1, -1, 0, 0}	{0, 1, 0, -1}	{1, 0, -1, 0}	$\{\frac{1}{2}, -\frac{1}{2}, 0, -\frac{1}{\sqrt{2}}\}$
C	{0, 0, 1, -1}	{1, 1, 1, -1}	{1, 1, 1, -1}	{1, 1, -1, 1}	{1, 1, 1, -1}	$\{0, -\frac{1}{\sqrt{2}}, \frac{1}{2}, \frac{1}{2}\}$
D	{1, 1, 1, -1}	{1, -1, -1, -1}	{1, 1, -1, 1}	{1, -1, -1, -1}	{1, -1, 1, 1}	$\{\frac{1}{2}, \frac{1}{2}, \frac{1}{\sqrt{2}}, 0\}$
E	{1, 0, -1, 0}	{1, 1, -1, 1}	{1, -1, -1, -1}	{1, -1, 1, 1}	{0, 0, 1, -1}	$\{\frac{1}{2}, -\frac{1}{2}, 0, \frac{1}{\sqrt{2}}\}$
F	{0, 1, 0, 1}	{0, 1, 0, -1}	{0, 1, 0, -1}	{0, 0, 1, -1}	{1, -1, -1, -1}	$\{0, \frac{1}{\sqrt{2}}, -\frac{1}{2}, \frac{1}{2}\}$
G	{1, -1, 1, 1}	{1, 0, 1, 0}	{1, 0, 1, 0}	{1, 1, 0, 0}	{1, 1, 0, 0}	$\{\frac{1}{\sqrt{2}}, 0, -\frac{1}{2}, -\frac{1}{2}\}$
H	{0, 0, 0, 1}	{1, 0, -1, 0}	{1, 0, -1, 0}	{1, -1, 0, 0}	{1, -1, 0, 0}	$\{\frac{1}{\sqrt{2}}, 0, \frac{1}{2}, \frac{1}{2}\}$
I	{0, 1, -1, 0}	{0, 1, 0, 0}	{0, 1, 0, 0}	{0, 0, 1, 0}	{0, 0, 1, 0}	$\{0, 0, -\frac{1}{\sqrt{2}}, \frac{1}{2}\}$
J		{1, 0, 0, 0}	{0, 0, 1, 0}	{0, 0, 1, 1}	{0, 1, 0, 0}	$\{0, 0, \frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\}$
K		{1, 1, -1, -1}	{1, 0, 0, 1}	{1, 0, 0, 1}	{1, 0, 0, 1}	$\{-\frac{1}{\sqrt{2}}, 0, -\frac{1}{2}, \frac{1}{2}\}$
L		{0, 1, 1, 0}	{1, 1, -1, -1}	{1, -1, 1, -1}	{1, 1, -1, 1}	$\{\frac{\sqrt{3}}{2}, -\frac{\sqrt{3}}{4}, -\frac{1}{4}, \frac{1}{4}\}$
M		{1, -1, 1, 1}	{0, 0, 1, -1}	{1, 0, -1, 0}	{1, -1, 1, -1}	$\{\frac{1}{2}, \frac{\sqrt{3}}{2\sqrt{2}}, \frac{\sqrt{3}}{4}, -\frac{\sqrt{3}}{4}\}$
N		{0, 1, 0, 1}	{0, 1, 0, 1}	{1, 1, 1, -1}	{0, 1, 0, -1}	
O				{0, 1, 0, 0}	{0, 0, 1, 1}	

LMNO, HIJK, DEFG, 9ABC, 5678, 1234, 4FGK, 78EG, BCDG, 3KNO, 68MO, ACMN, 12IJ, 258J, 29CI.

LMNO, HIJK, DEFG, 9ABC, 5678, 34KO, 78JN, BCIN, 68FG, ACEG, 12HM, 1234, DGHO, 2458, 149C.

LMNO, IJKO, EFGH, ABCD, 6789, 345K, 125N, 289J, 1CDJ, 479M, 3BDM, 69GH, ADFH, 5EHO, 5KNO.

24-16(2,3) do not contain (c):

LMNO, HIJK, DEFG, 9ABC, 5678, 8BCO, 7FGO, ACJK, EGIK, 1234, 346N, 249C, 14DG, 5HKN, 125M, 56MN.

LMNO, HIJK, DEFG, 9ABC, 5678, BCJK, FGIK, ACNO, EGMO, 3478, 1256, 48HK, 269C, 16DG, 38LO, 1234.

LMNO, HIJK, DEFG, 9ABC, 678C, FGJK, EGNO, 3458, 125B, 5CDG, 1267, 349A, 27IK, 4AHK, 17MO, 3ALO.

LMNO, HIJK, DEFG, 9ABC, BCJK, FGIK, 5678, 3478, 1256, 1234, ACNO, EGMO, 489C, 47DG, 26HK, 16MN.

LMNO, HIJK, DEFG, 9ABC, CFGO, EGJK, 5678, 3478, 1256, 1234, 24BO, 9AMN, 68DG, 14IK, AHKN, 67AM.

24-17 contain both (c) and (d):

LMNO, HIJK, DEFG, FGJK, EGNO, 9ABC, 5678, 3478, 24BC, 68IK, ACHK, 58MO, 9CLO, 14DG, 1256, 139A, 1234.

LMNO, HIJK, JKNO, DEFG, 9ABC, 78BC, 56FG, 349A, 12DE, 3478, 1256, ACIK, EGHK, 48MO, 26LO, 259C, 47DG.

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